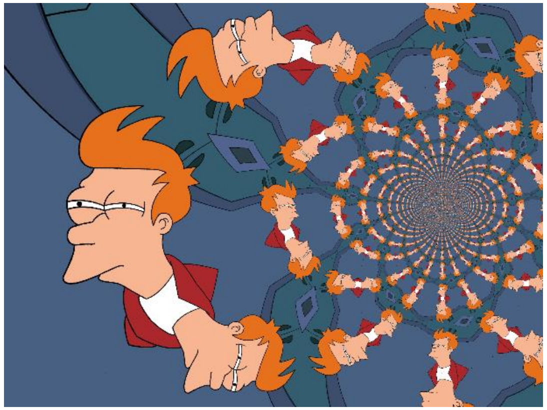
What to do
When all you have is
Brownian motion

Student probability day VII,
16/05/19



Renan Gross, WIS

Conformal mapping

Warning

This presentation shows graphical images of Brownian motion.

Viewer discretion is advised.

The Skorokhod embedding problem

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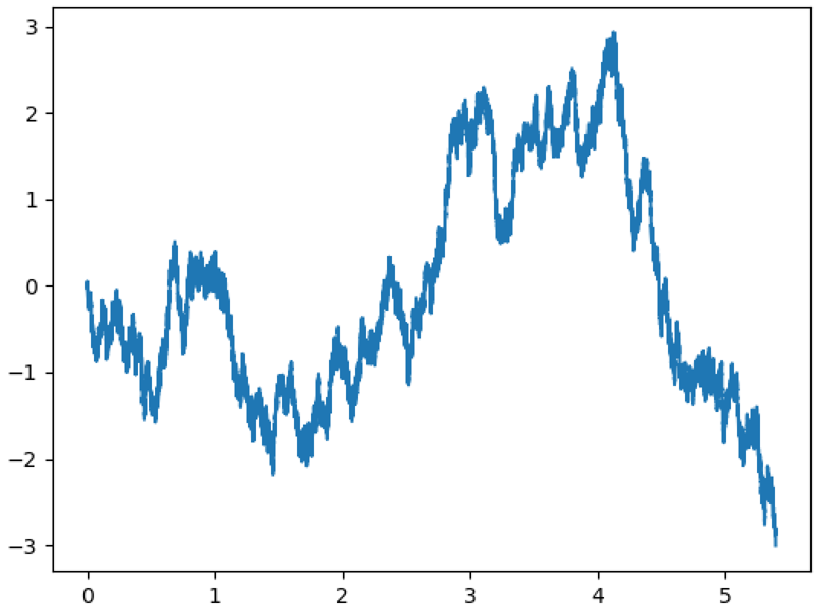
- Life gives you Brownian motion.



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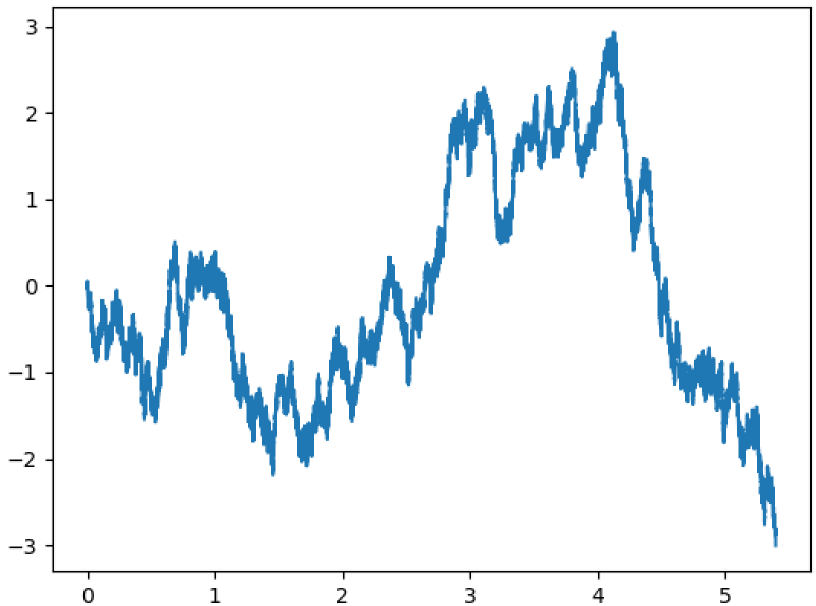
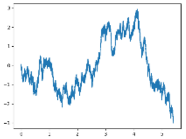


- But you do not want Brownian motion. You want to sample from a distribution μ.
- How do you sample from μ using your Brownian motion?

The Skorokhod embedding problem



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- If you choose T in a special way, then perhaps B_T distributes as μ?

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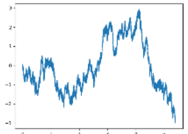

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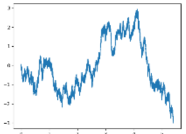

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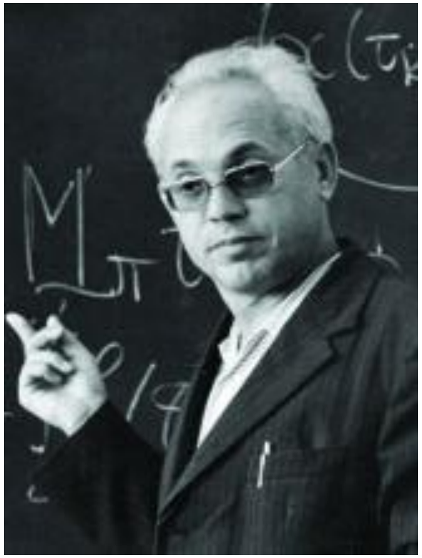
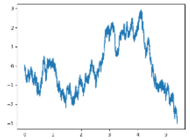


(This is Skorokhod)

Examples



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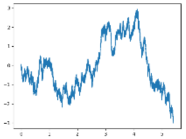

- **Gaussian distribution**: Brownian motion is just a fancy name for Gaussians: $\boldsymbol{B_t \sim N(0, t)}$.

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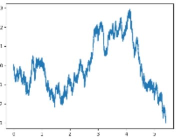

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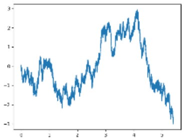

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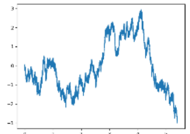

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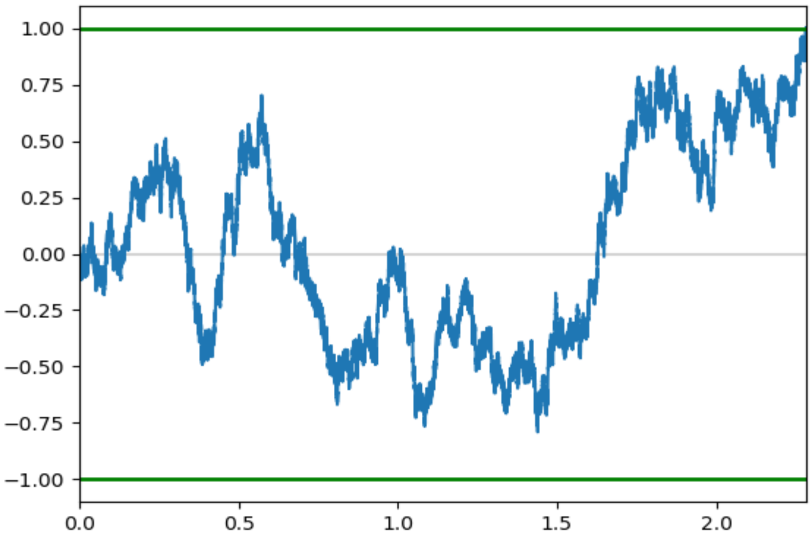

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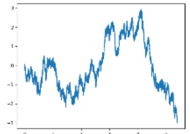

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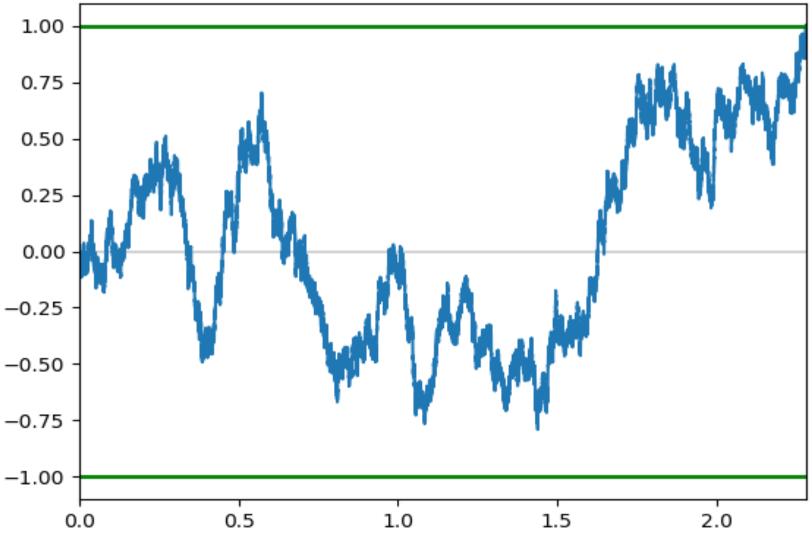
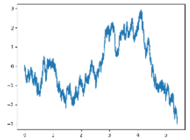


Uniform distribution on $[-1,1]$?

Theorem statement



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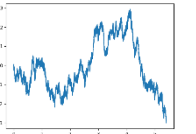

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Theorem: Let μ have 0 mean and finite variance. Then there exists a stopping time T with $\mathbb{E}T < \infty$ such that $B_T \sim \mu$.

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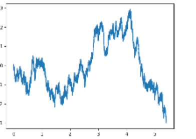

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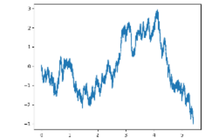

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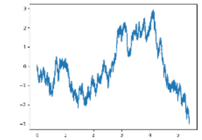

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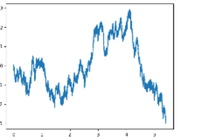

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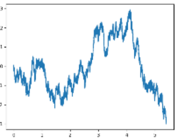

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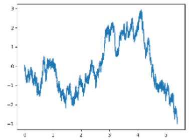

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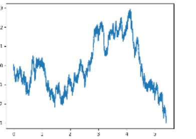

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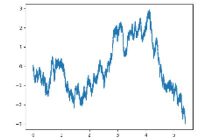

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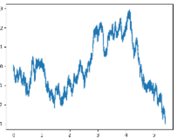

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- Also Gross 19.

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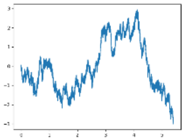

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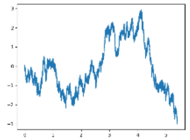

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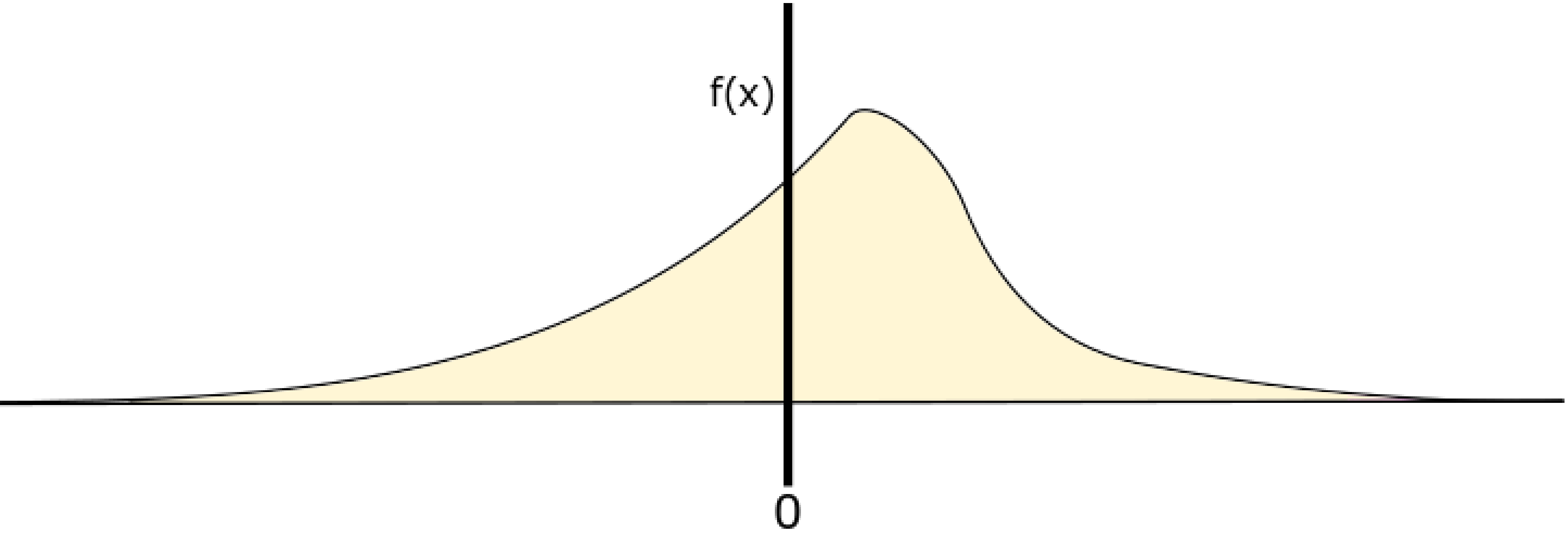

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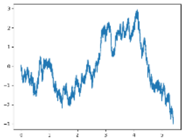

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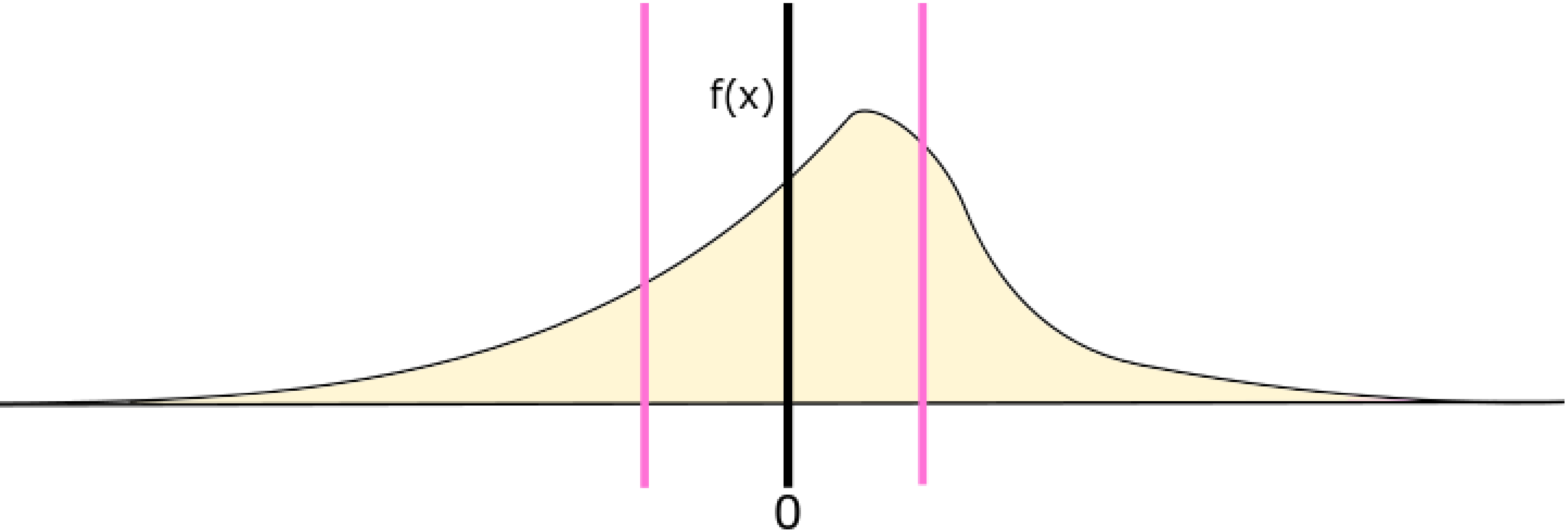

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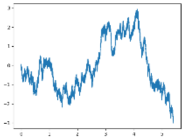

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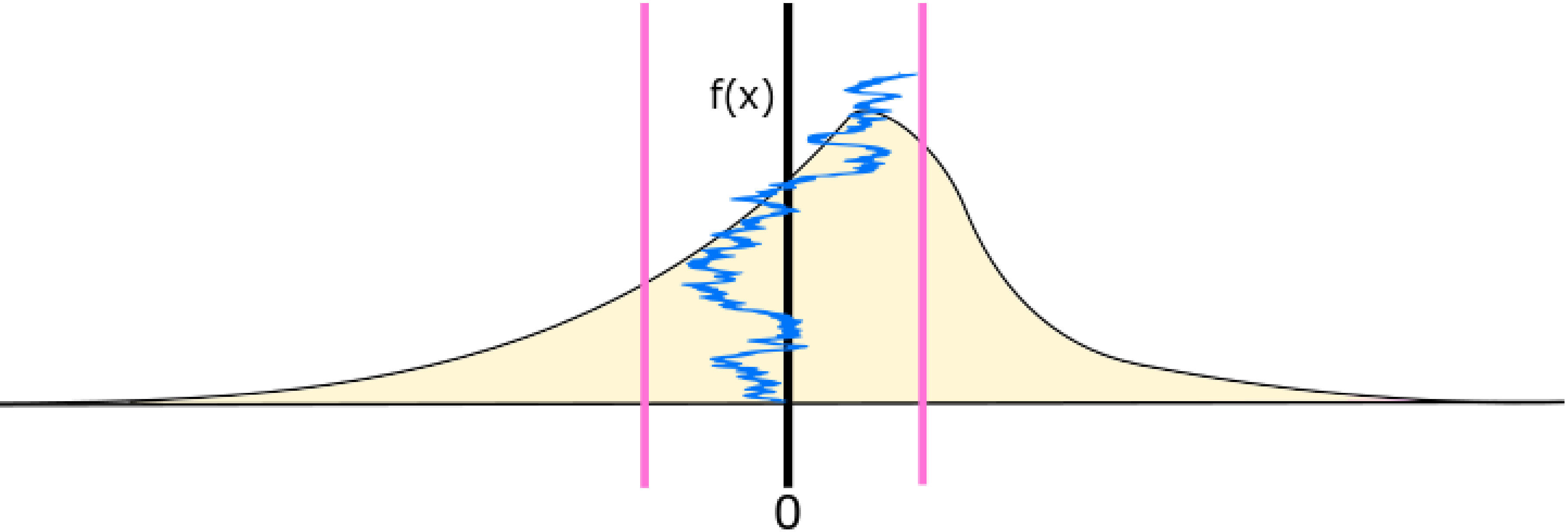

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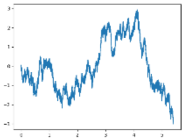

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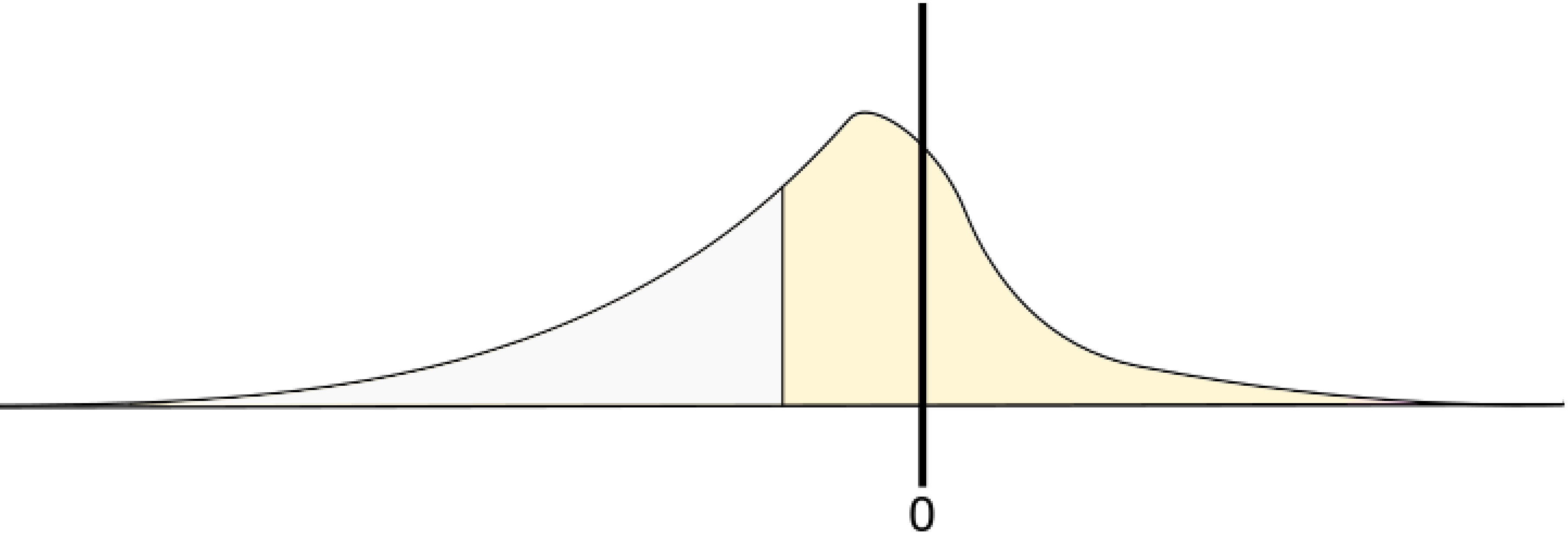

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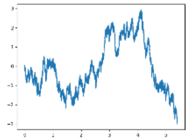

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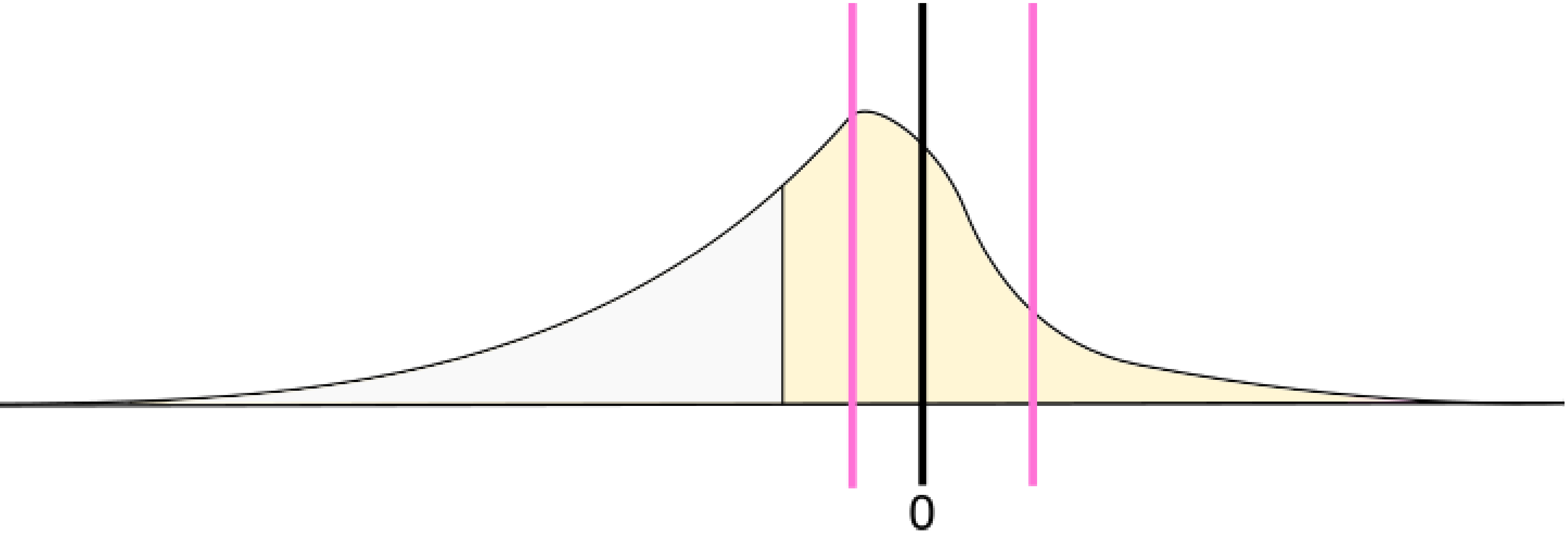

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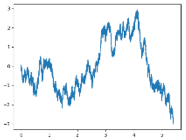

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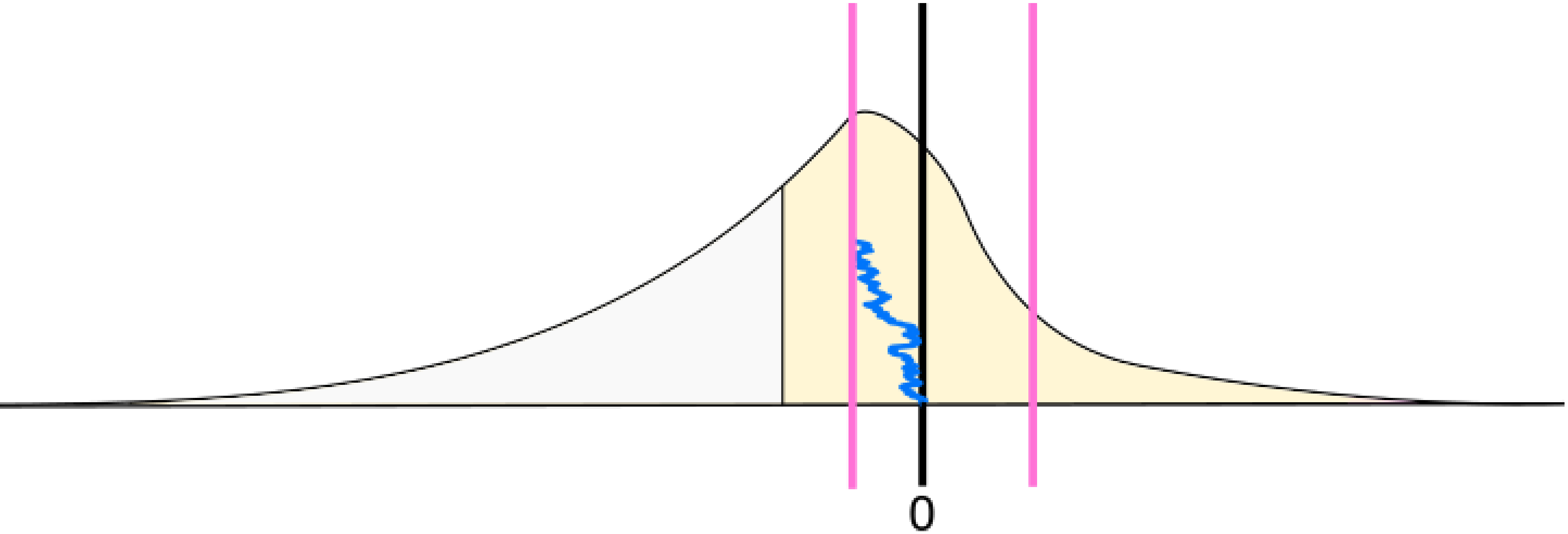

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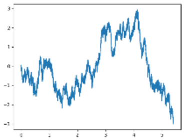

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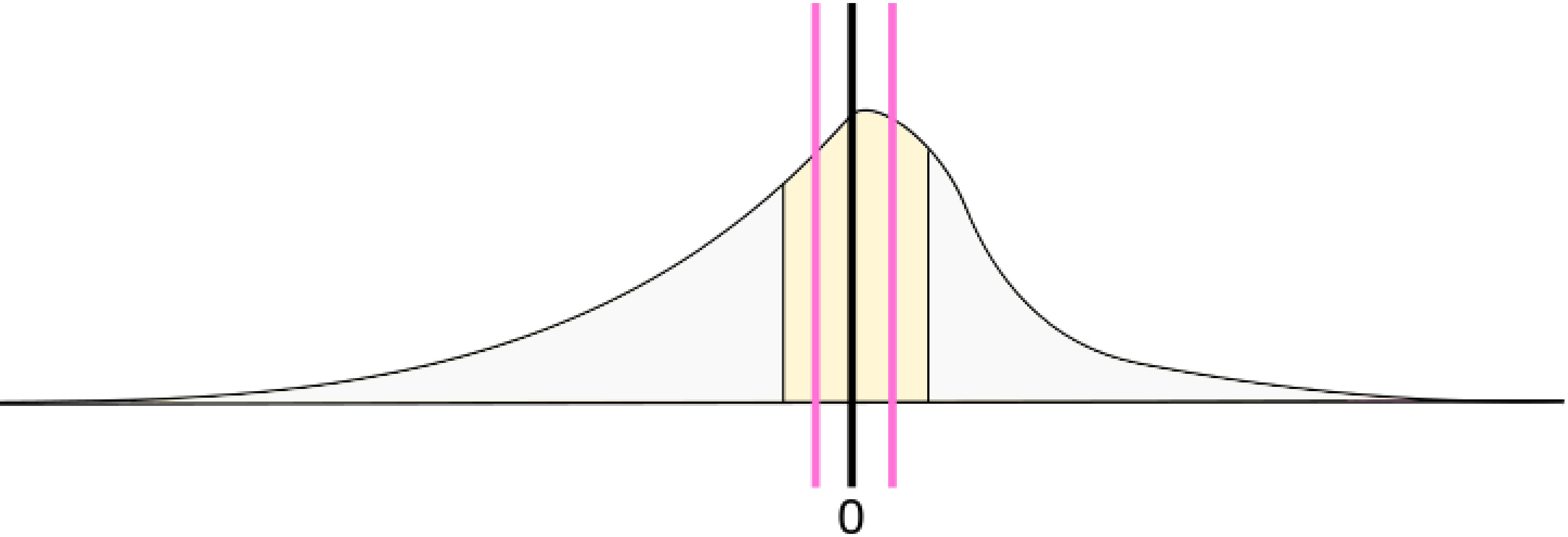

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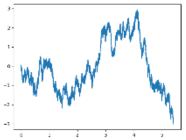

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- The hitting time of the graph (X_t, t) with some barrier.

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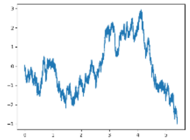

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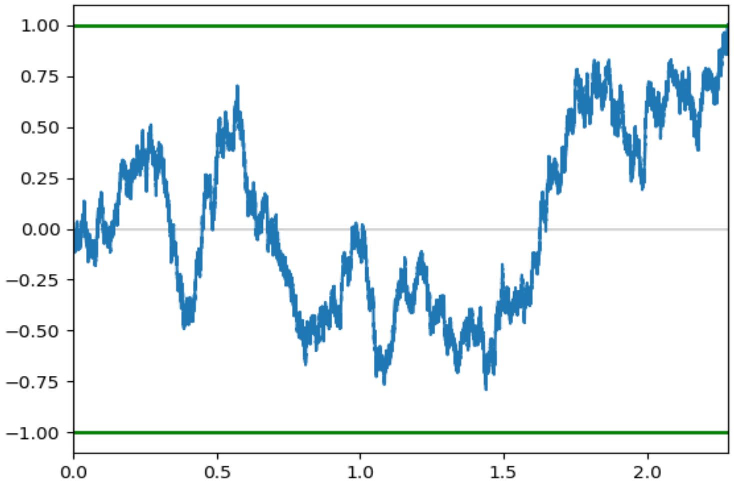

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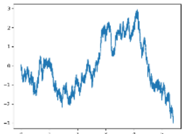

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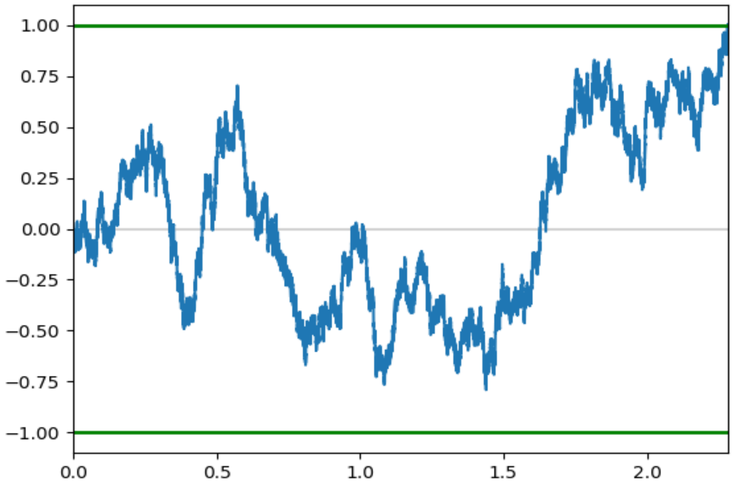

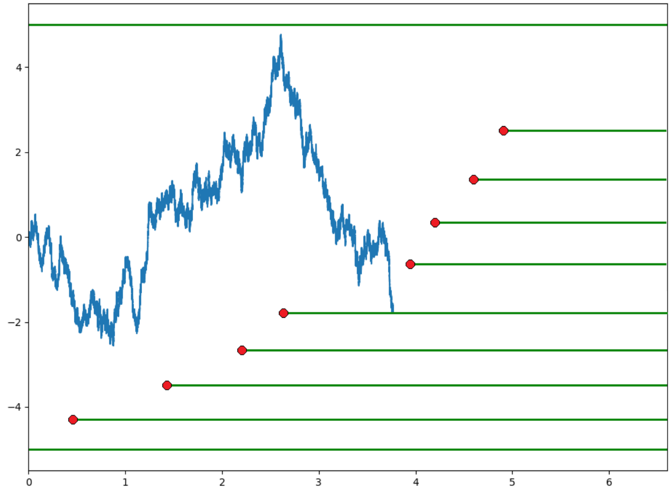

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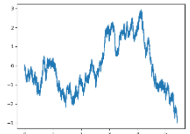

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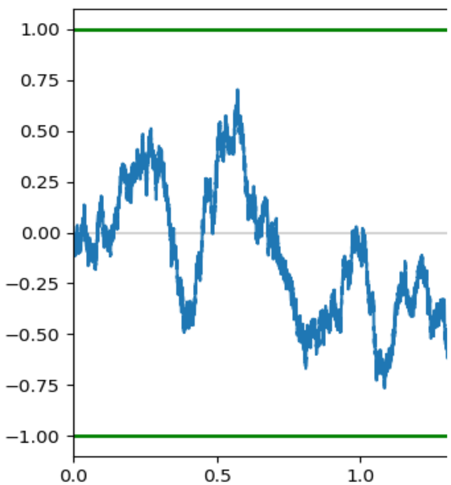

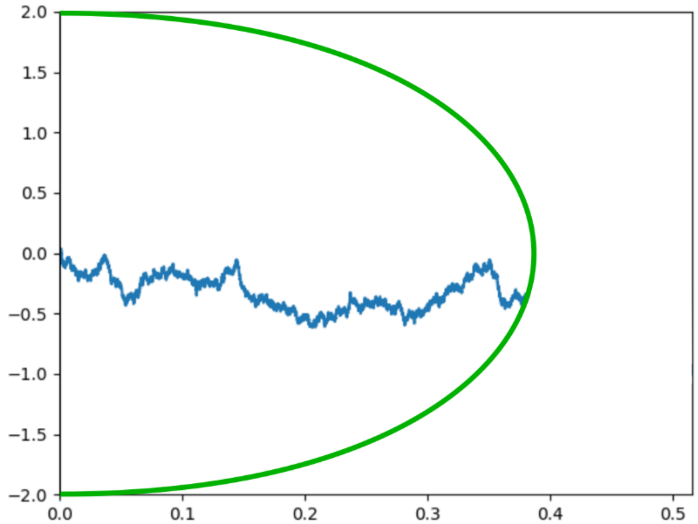

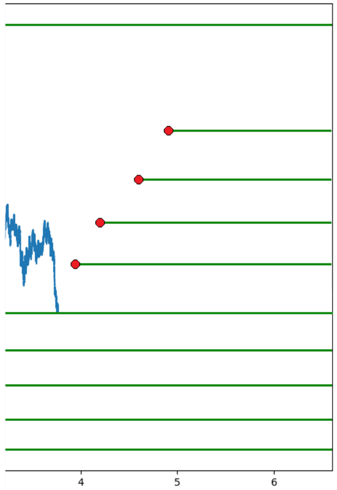

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- Rather, uses conformal mappings

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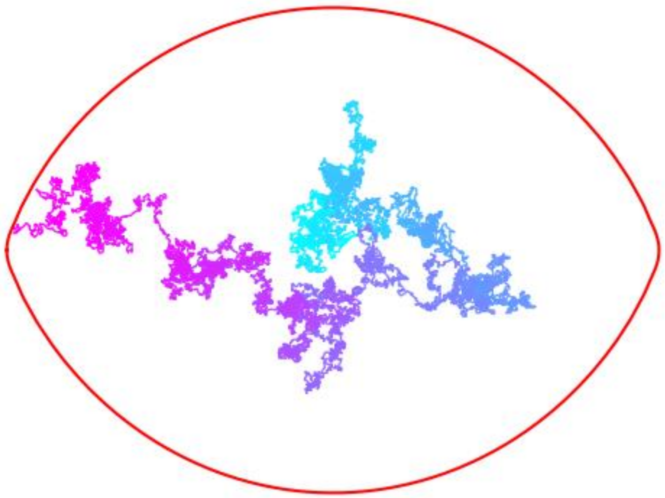
Theorem: Let μ have 0 mean and finite variance. Let B_t be a planar Brownian motion. There exists a simply connected domain Ω such that when B_t exits Ω, its x coordinate distributes as μ.



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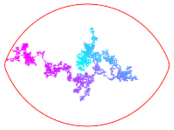

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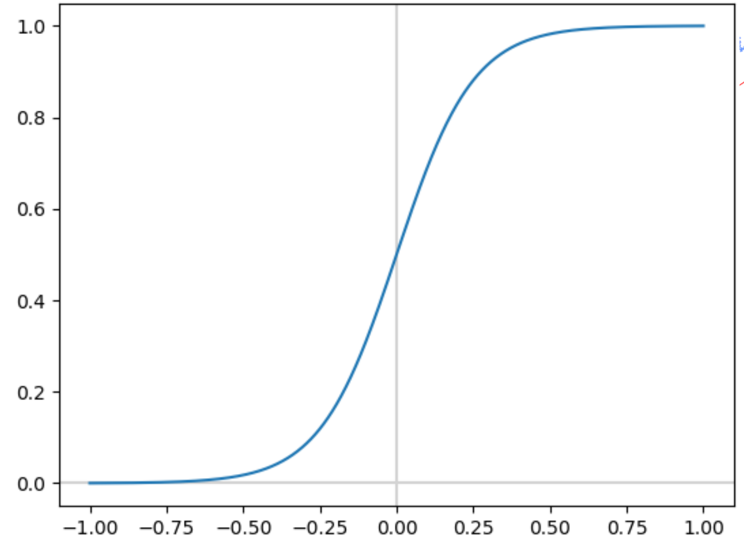

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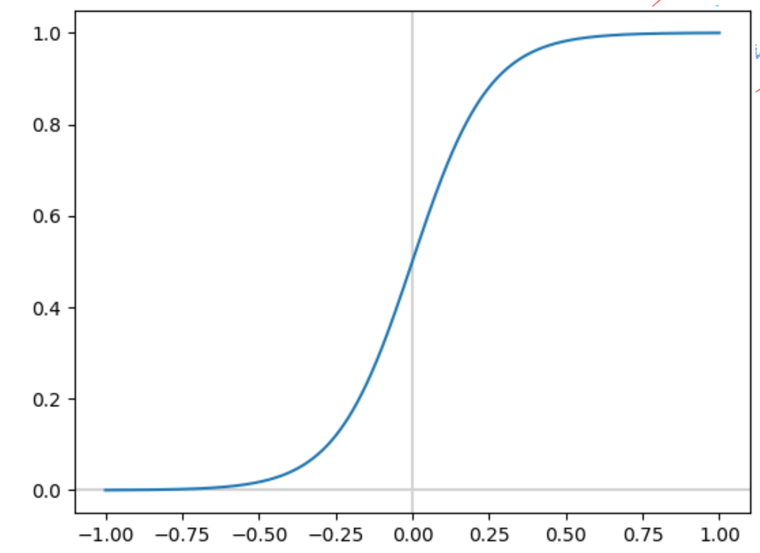
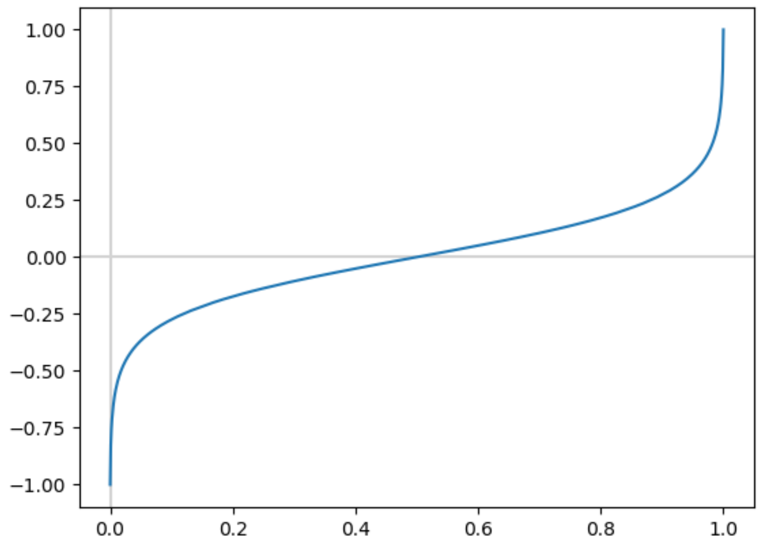
- Here is its inverse cdf, F_μ^{-1}.

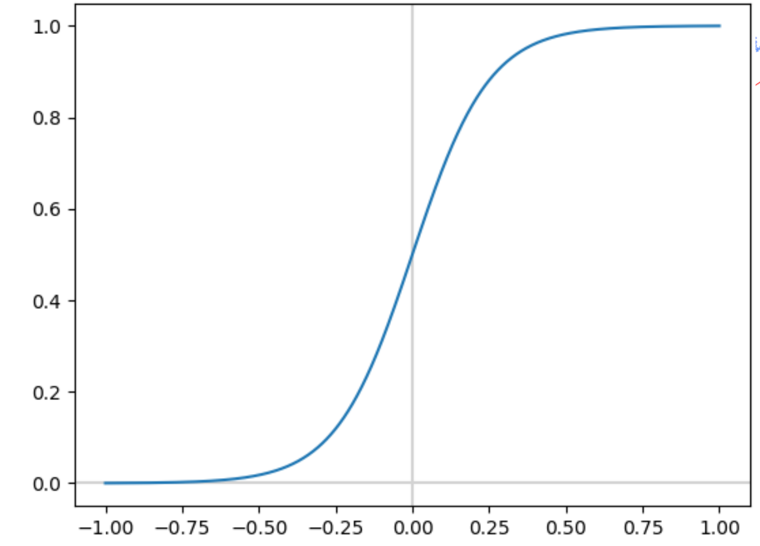


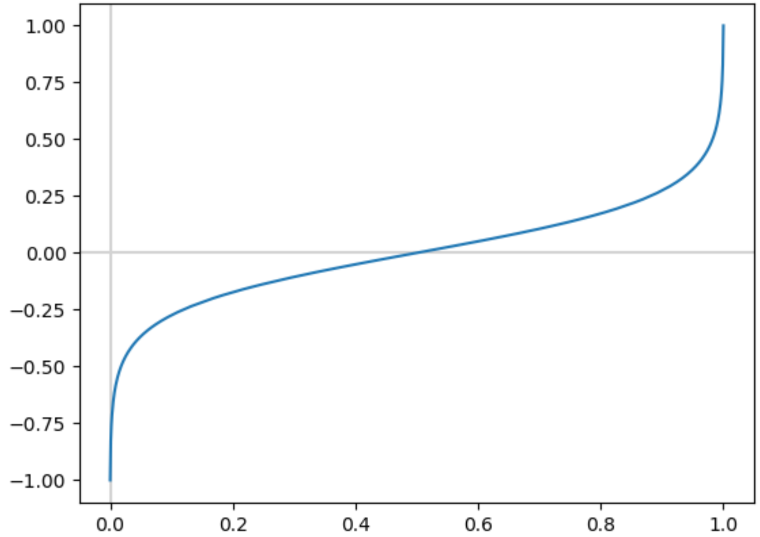


A conformal solution

- The idea is simple, actually.
- Here is the cdf of mu, F_μ.
- Here is its inverse cdf, F_μ^{-1}.
- The inverse mapping theorem says that if U is uniform, then

$$F_\mu^{-1}(U) \sim \mu$$





[[Any T satisfying $T(U) \sim \mu$ must satisfy

$$F_\mu(x) = \mathbb{P}[X \leq x] = \mathbb{P}[T(U) \leq x] = \mathbb{P}[U \leq T^{-1}(x)] = T^{-1}(x)]]$$

A conformal solution

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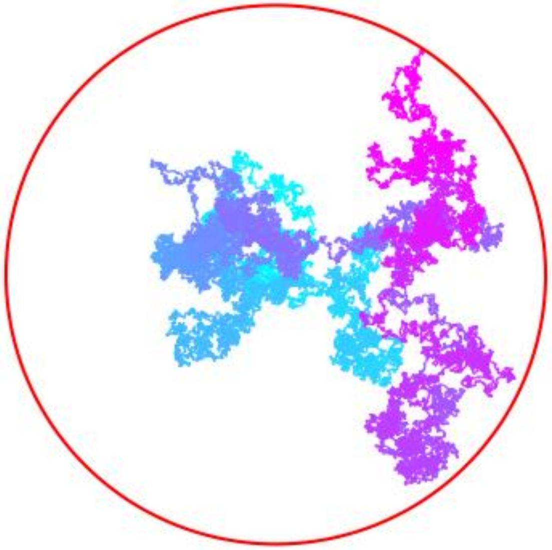
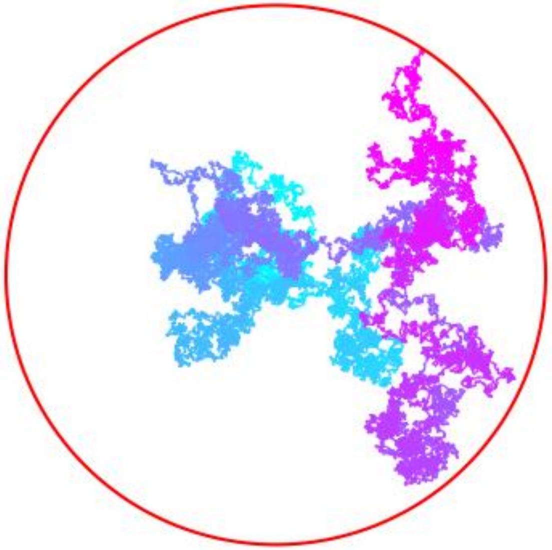
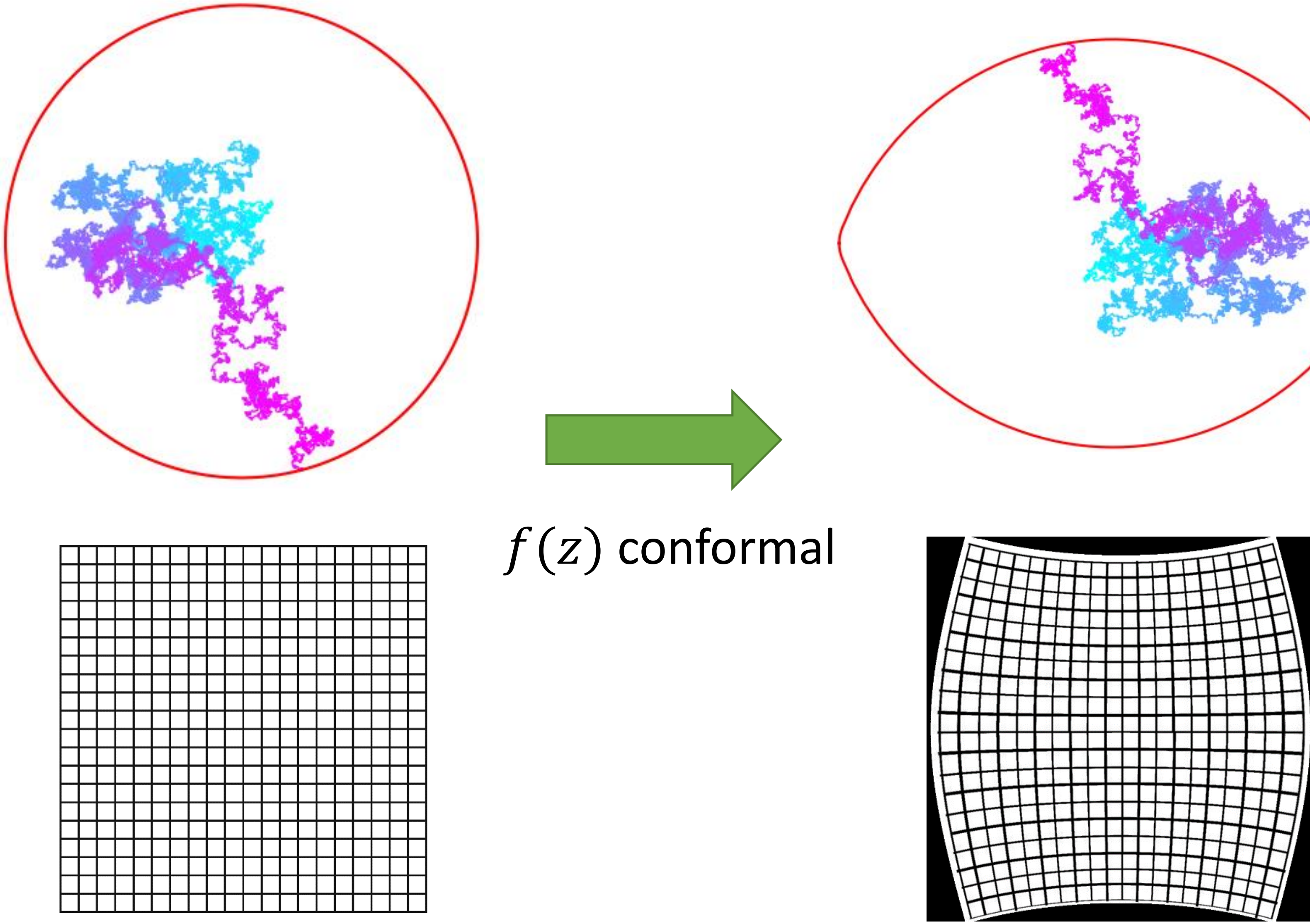


A conformal solution

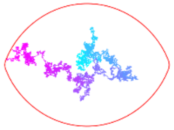


- If only there was some way of using B_t to sample F_μ^{-1} uniformly!
- How convenient! Brownian motion is uniform on the circle
- How convenient! Brownian motion is **conformally invariant**
- That is, the image of Brownian motion under a conformal map is a (time-changed) Brownian motion as well


$f(z)$ conformal

A conformal solution

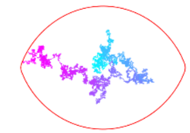
A conformal solution



- Since $\arg(B_{T_{circle}})$ is uniform, all we need is a conformal map ψ which has

$$\mathrm{Re}\{\psi(e^{i\theta})\} = F_{\mu}^{-1}(\theta)$$

A conformal solution



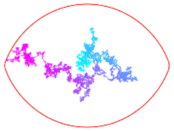
- Since $\arg\left(B_{T_{circle}}\right)$ is uniform, all we need is a conformal map ψ which has

$$\operatorname{Re}\left\{\psi\left(e^{i\theta}\right)\right\} = F_{\mu}^{-1}(\theta)$$

- Luckily for us, on the unit circle, a Fourier series and a power series are the same thing.

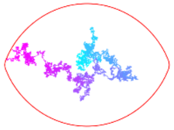
$$f(z) = \sum_{n=0}^{\infty} a_n z^n = \sum_{i=0}^{\infty} a_n e^{in\theta} = \sum_{i=0}^{\infty} a_n (\cos n\theta + i \sin n\theta\ )$$

A conformal solution



- Let $\varphi_\mu(\theta) = F_\mu^{-1}(\frac{|\theta|}{\pi})$

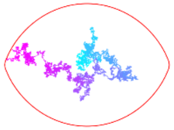
A conformal solution



- Let $\varphi_\mu(\theta) = F_\mu^{-1}(\frac{|\theta|}{\pi})$
- Expand $\varphi_\mu(\theta)$ as an even Fourier series:

$$\varphi_\mu(\theta) = \sum_{n=0}^{\infty} \widehat{\varphi_\mu}(n) \cos n\theta$$

A conformal solution



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- On the unit circle, the x-coordinate (i.e real part) of ψ_μ agrees with φ_μ

$$Re\{\psi_\mu(e^{i\theta})\} = \sum_{n=0}^{\infty} \widehat{\varphi_\mu}(n) \cos n\theta = \varphi_\mu(\theta)$$

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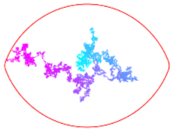

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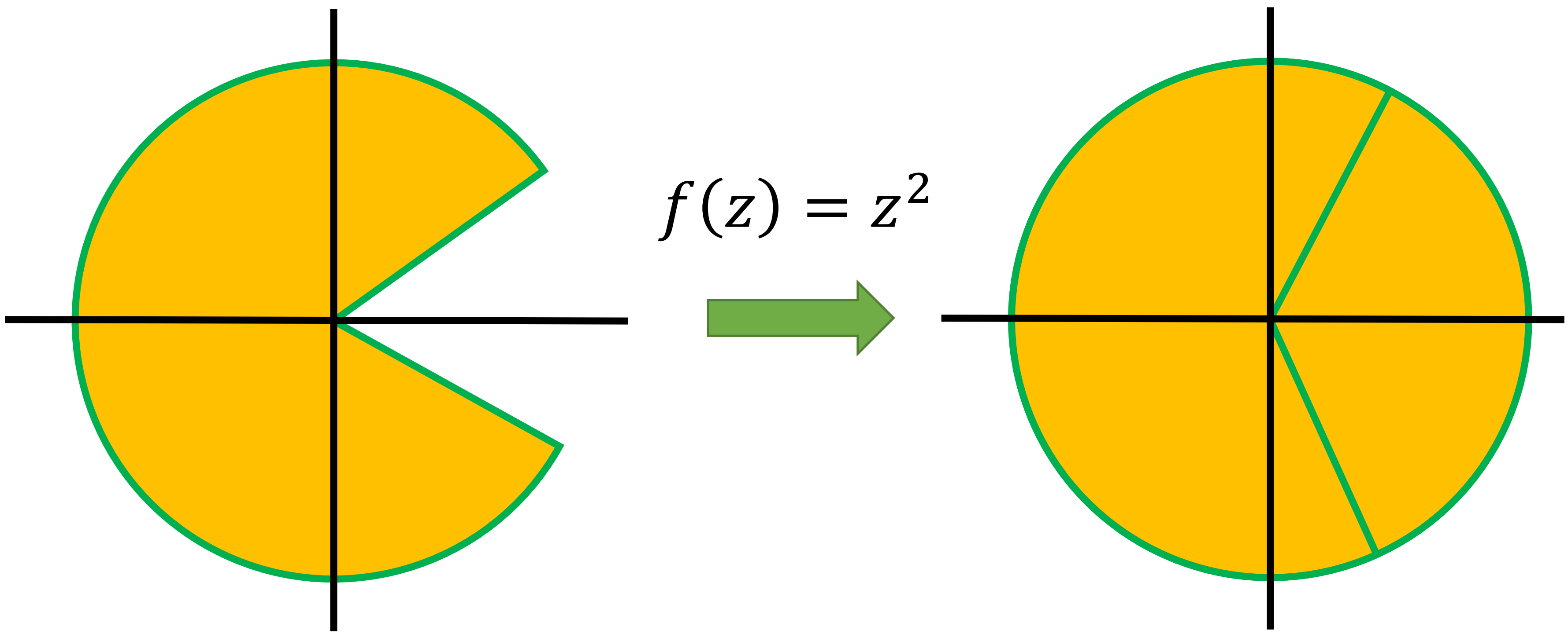
- The Fourier coefficients decay, so ψ_μ is analytic inside the unit disc

A potential problem

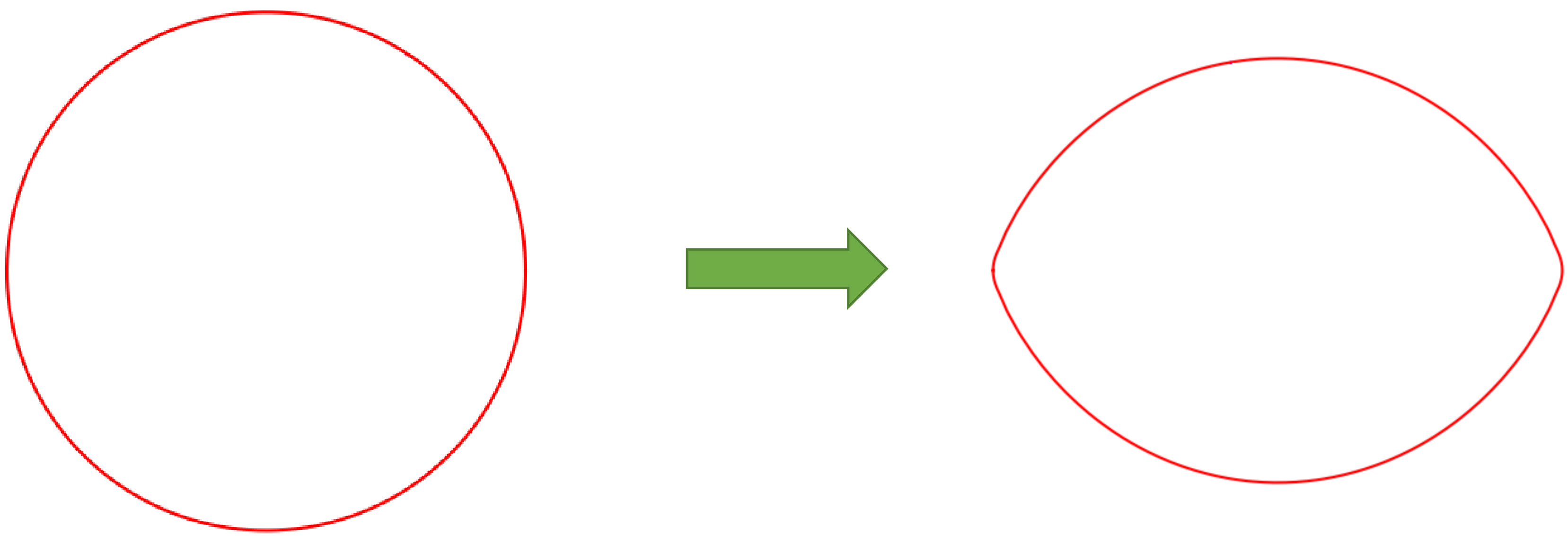
- Even though Brownian motion is preserved under analytic maps, in order to transform boundary to boundary we must be one-to-one
- Otherwise:



$$f(z) = z^2$$

A potential problem

- For “nice” enough μ, this is not a problem
- If μ is bounded and F_μ is strictly monotone increasing then F_μ^{-1} is continuous and bounded.
- ψ_μ then maps the circle’s boundary to a simple closed loop, which is the boundary of Ω



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- For nasty μ, we don't have that luxury
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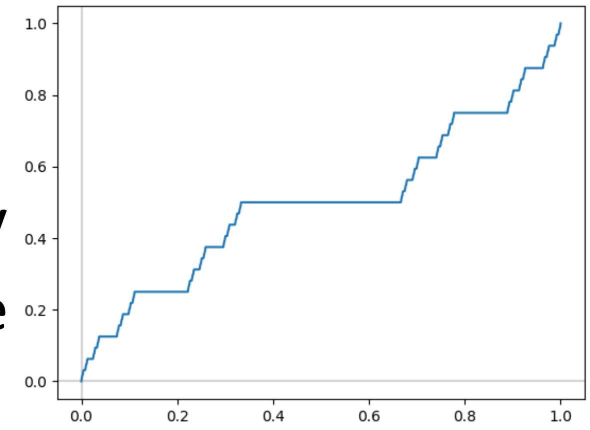
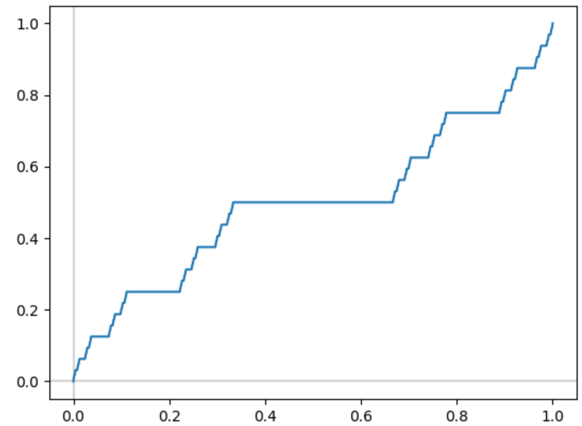
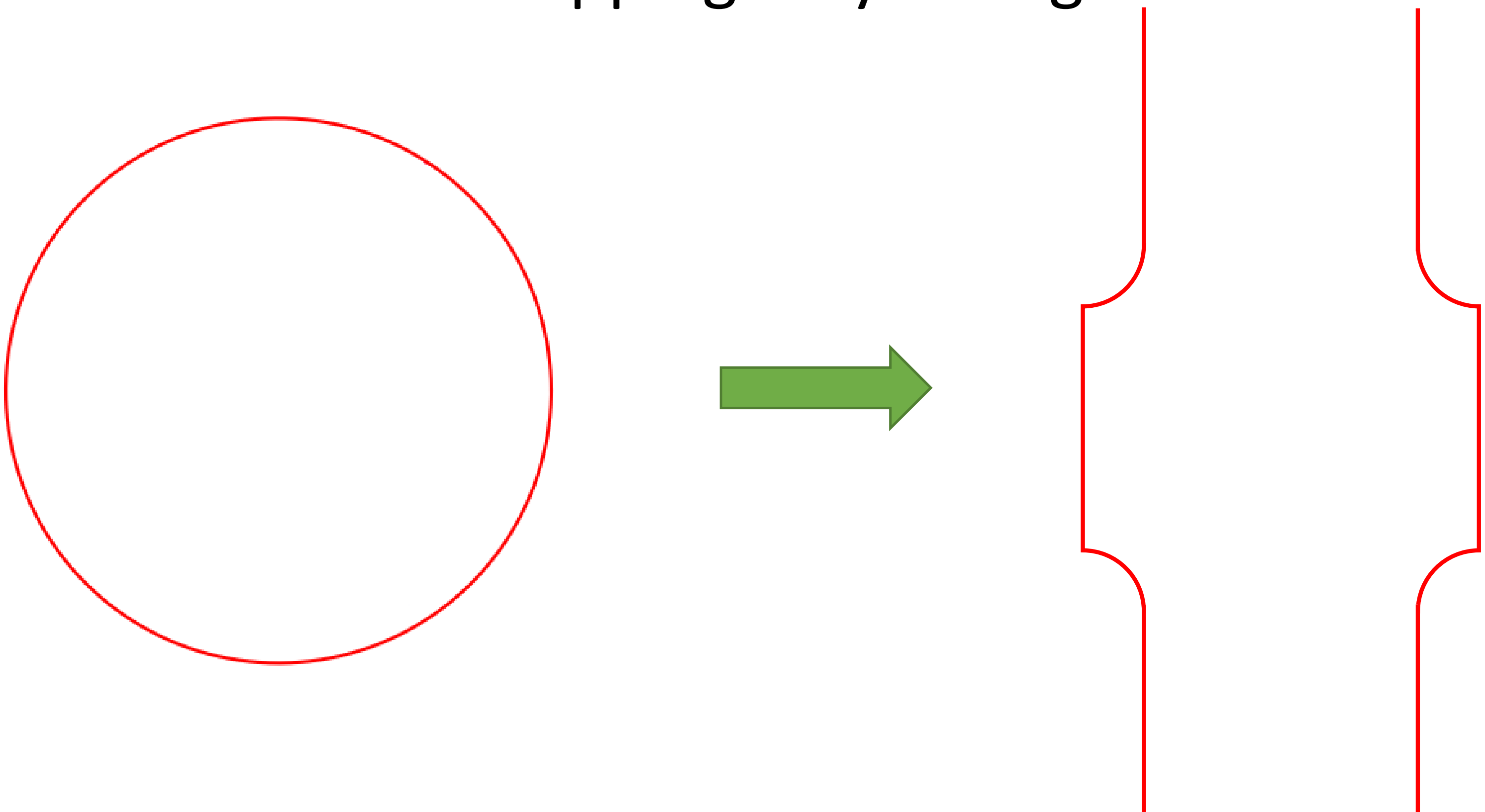


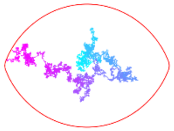
A potential problem

- For nasty μ, we don’t have that luxury
  - E.g: an atomic distribution with finite we rational)
  - E.g: The Cantor distribution
- In this case the mapping may diverge





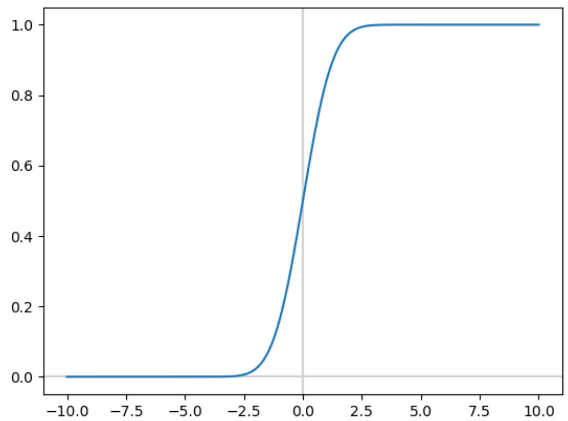
A solution

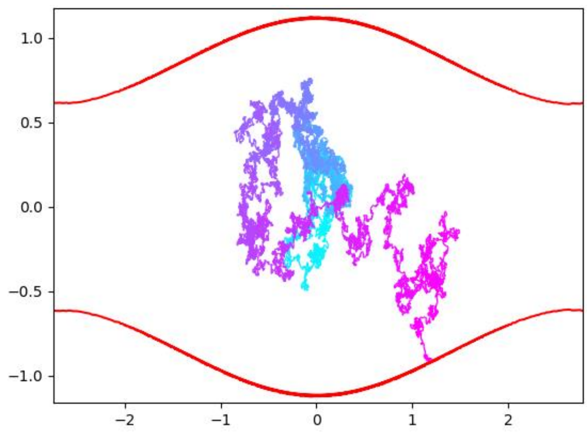


- **Theorem**: Let $\{f_k(z)\}$ be a series of one-to-one functions on a domain D which converge uniformly on every compact subset of D to a function f. Then f is either one-to-one or constant.
- If we take nice smooth functions F_k (not necessarily CDFs) which converge to F_k, we'll get ψ_ks which are one-to-one and which will converge to ψ_μ.

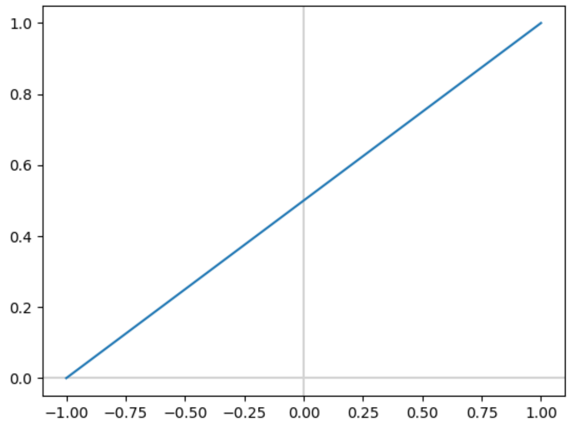
Examples

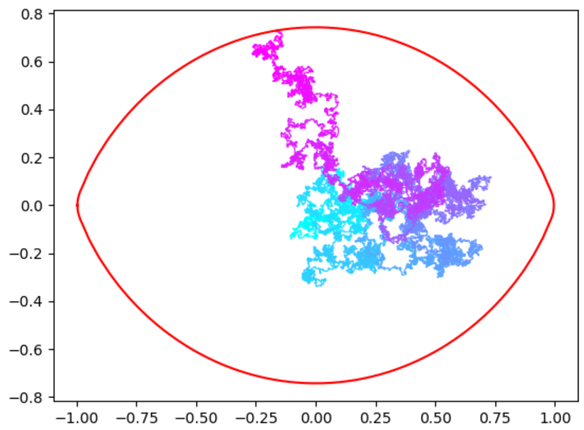
Gaussian





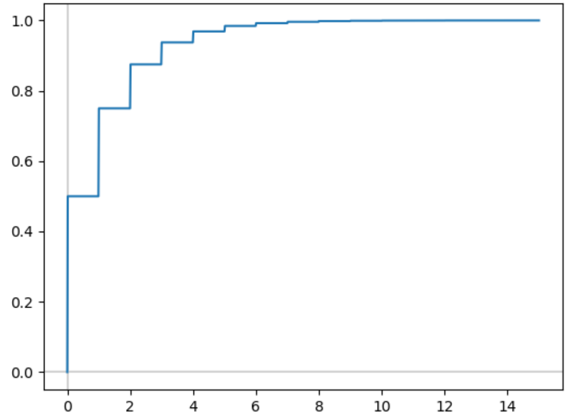
Uniform

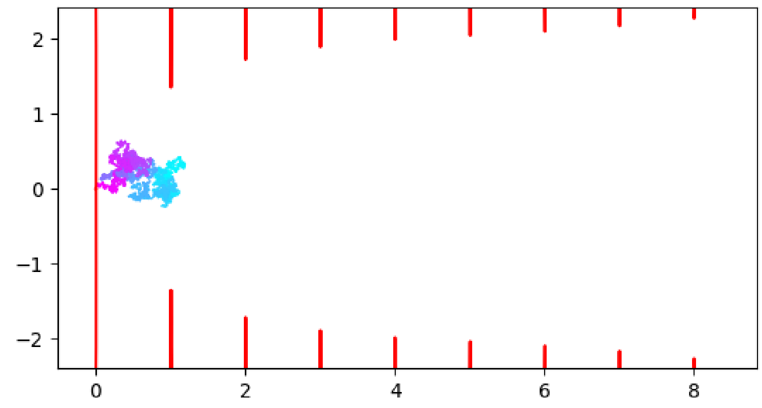




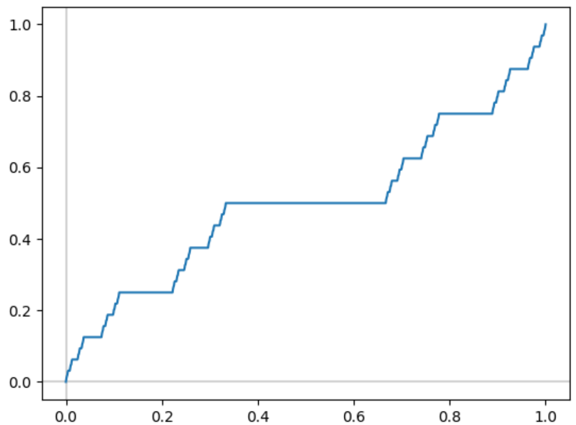
Examples

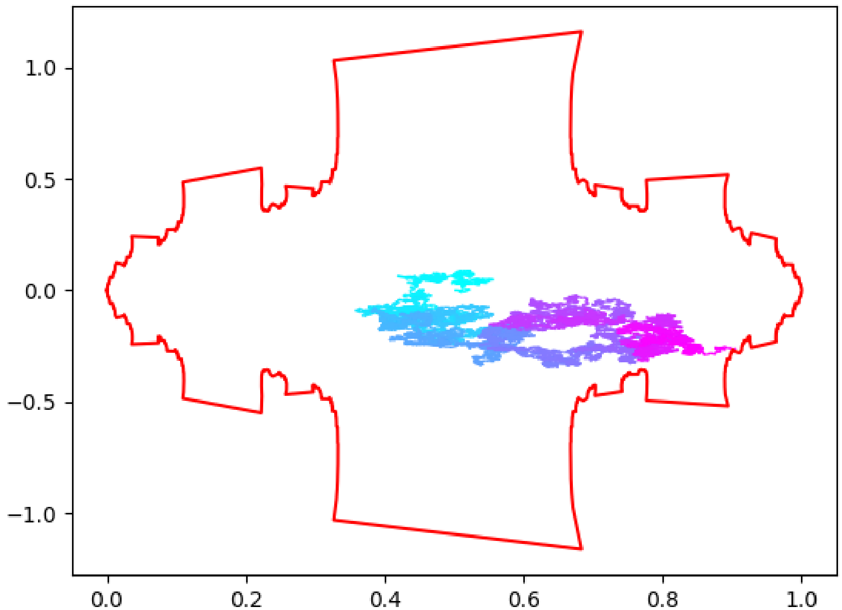
$P[k] \sim \frac{1}{2^k}$





Cantor





For more information, call

1-800-https://arxiv.org/abs/1905.00852

Thanks!