Noise sensitivity from fractional query algorithms

Renan Gross, Weizmann Institute of Science

Boolean functions

A Boolean function is a function $f: \{-1,1\}^n \to \{-1,1\}$.

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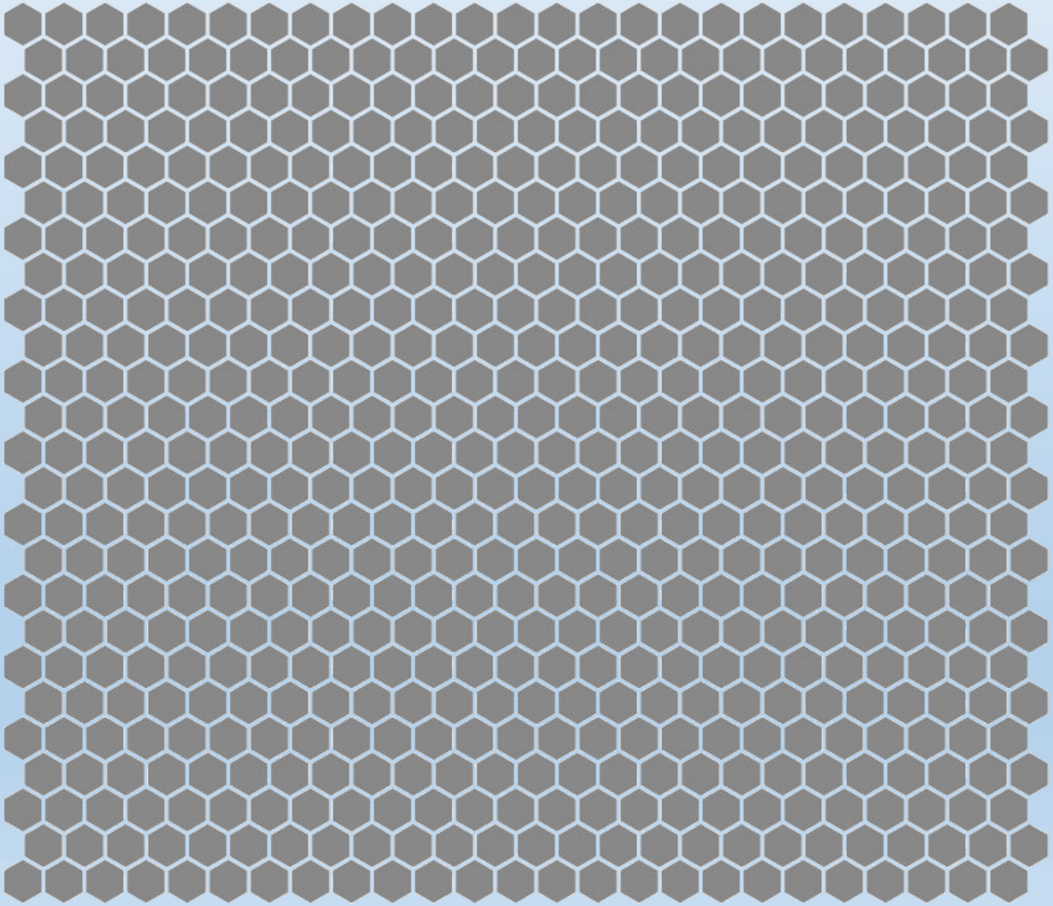
Example: Majority

$$f(x) = \text{sign} \sum_{i=1}^{n} x_i$$

Percolation

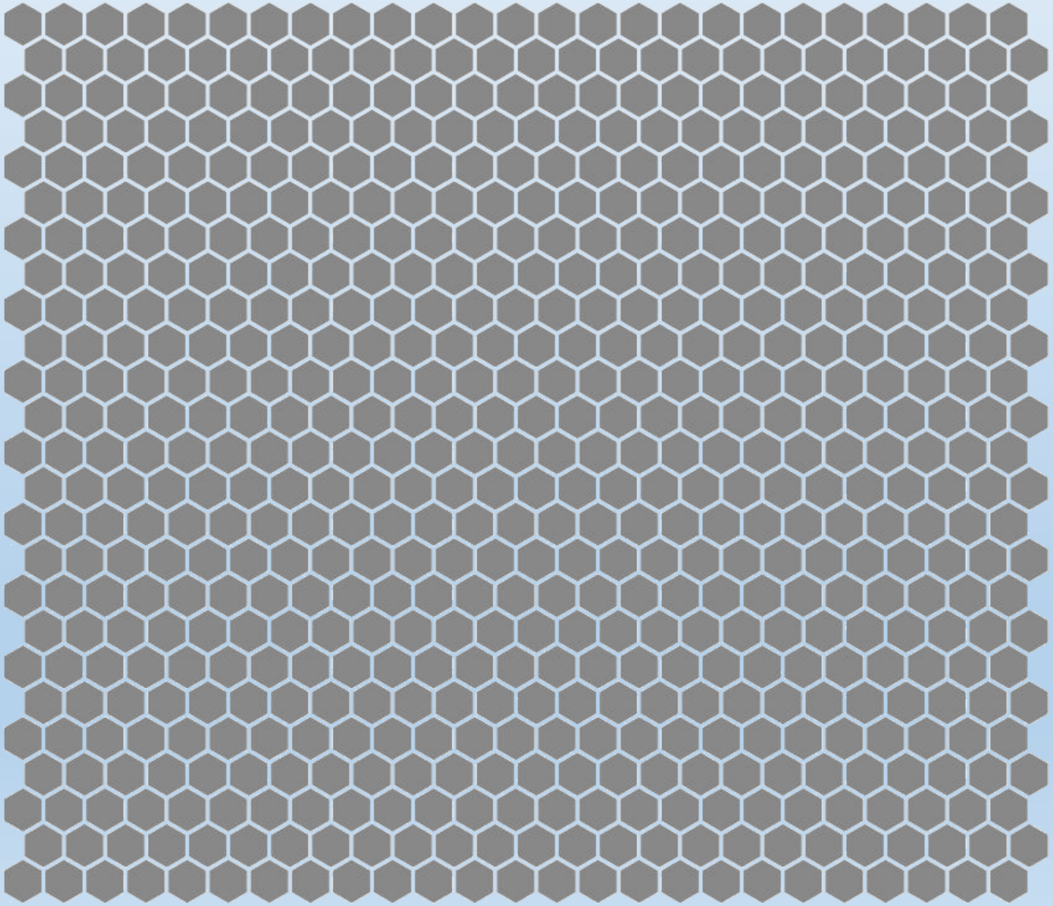
$f:\{-1,1\}^n \to \{-1,1\}$



Percolation

$f: \{-1,1\}^n \to \{-1,1\}$

$$x = 1, 1, -1, 1, -1, -1, -1, -1, -1, 1, 1, -1, 1, -1, \ldots$$



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$f\colon \{-1,1\}^n \rightarrow \{-1,1\}$

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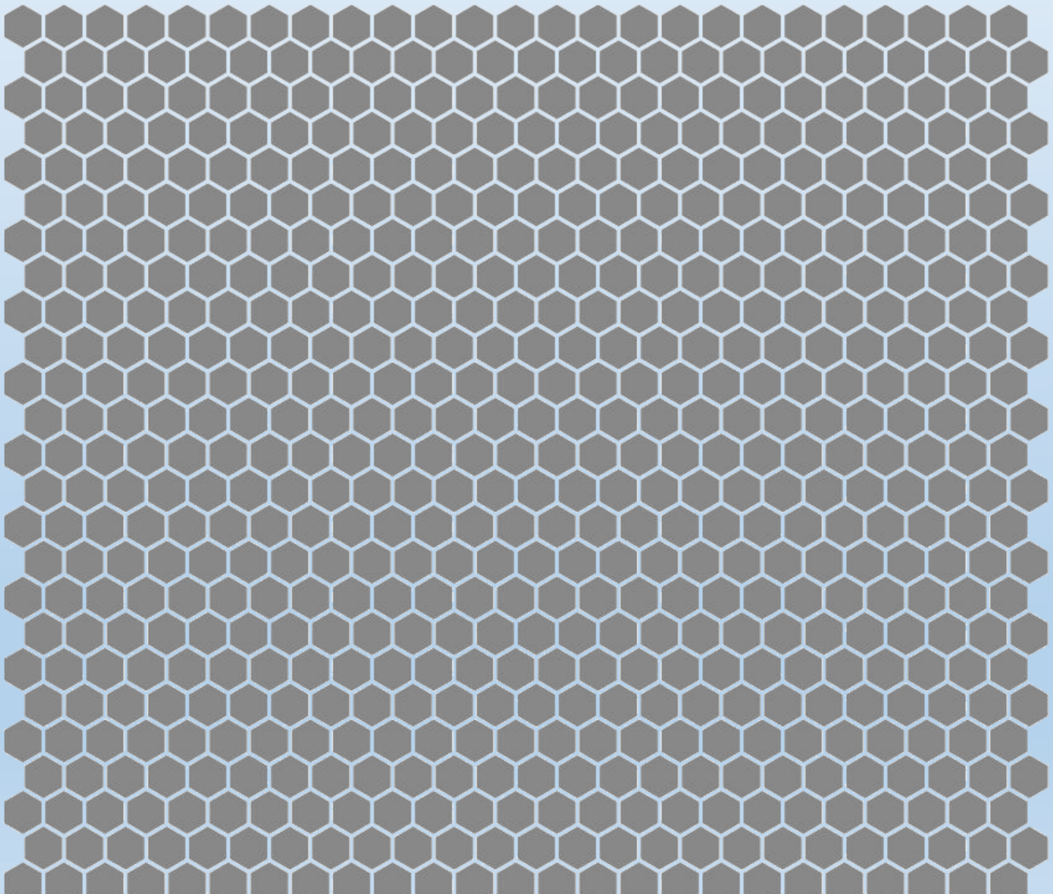
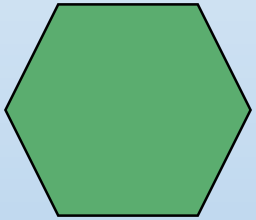
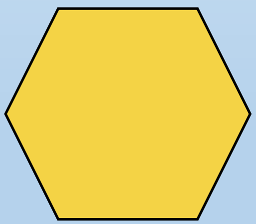


$1 =$ 

$-1 =$ 

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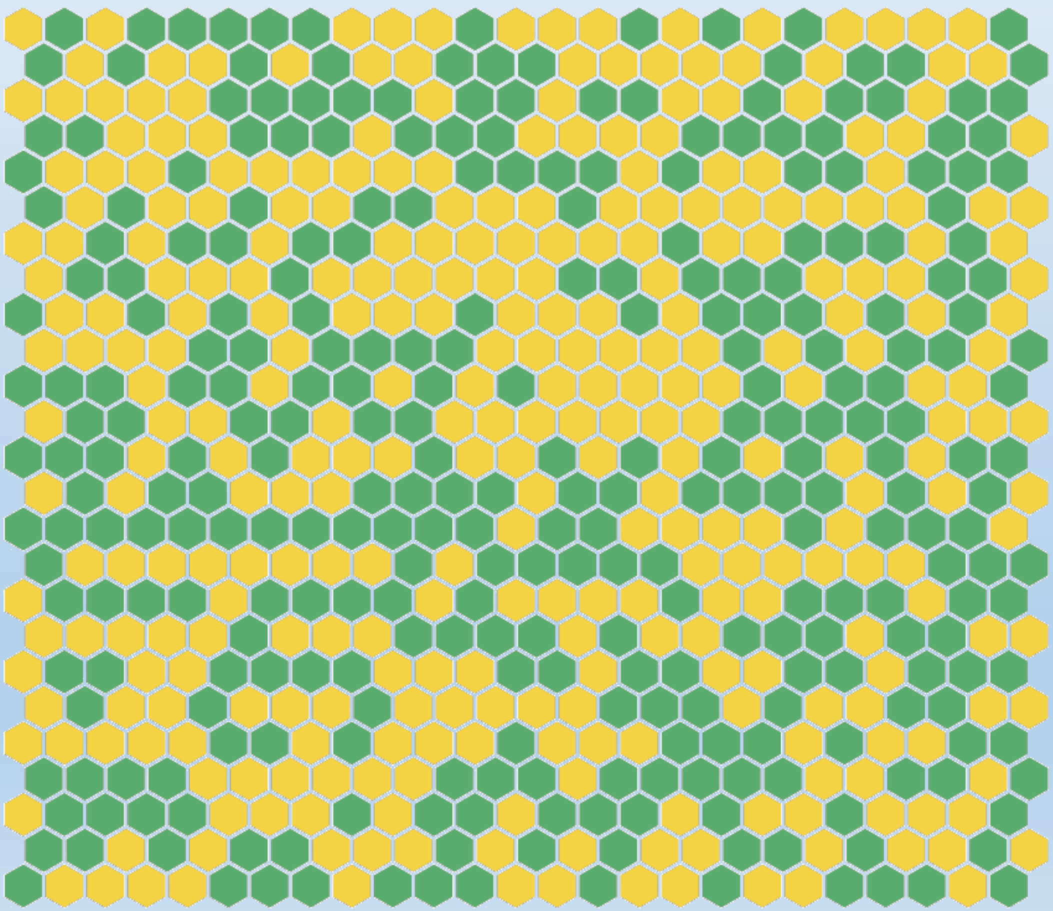
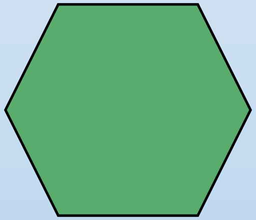
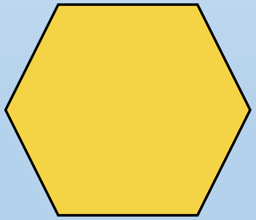


$1 =$ 

$-1 =$ 

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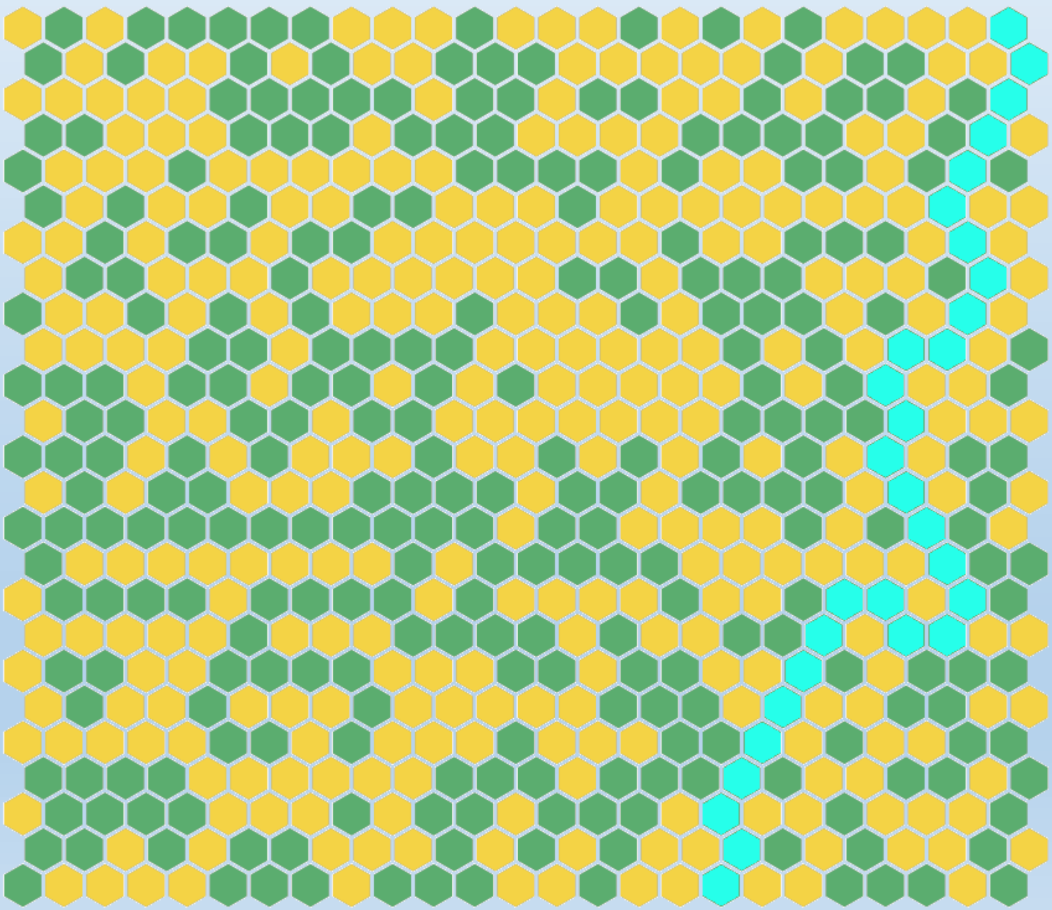
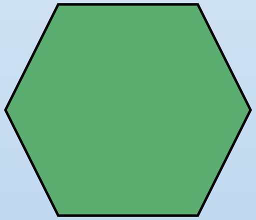
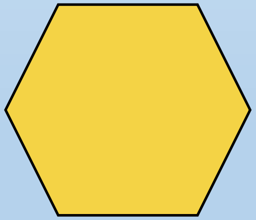


$1 =$ 

$-1 =$ 

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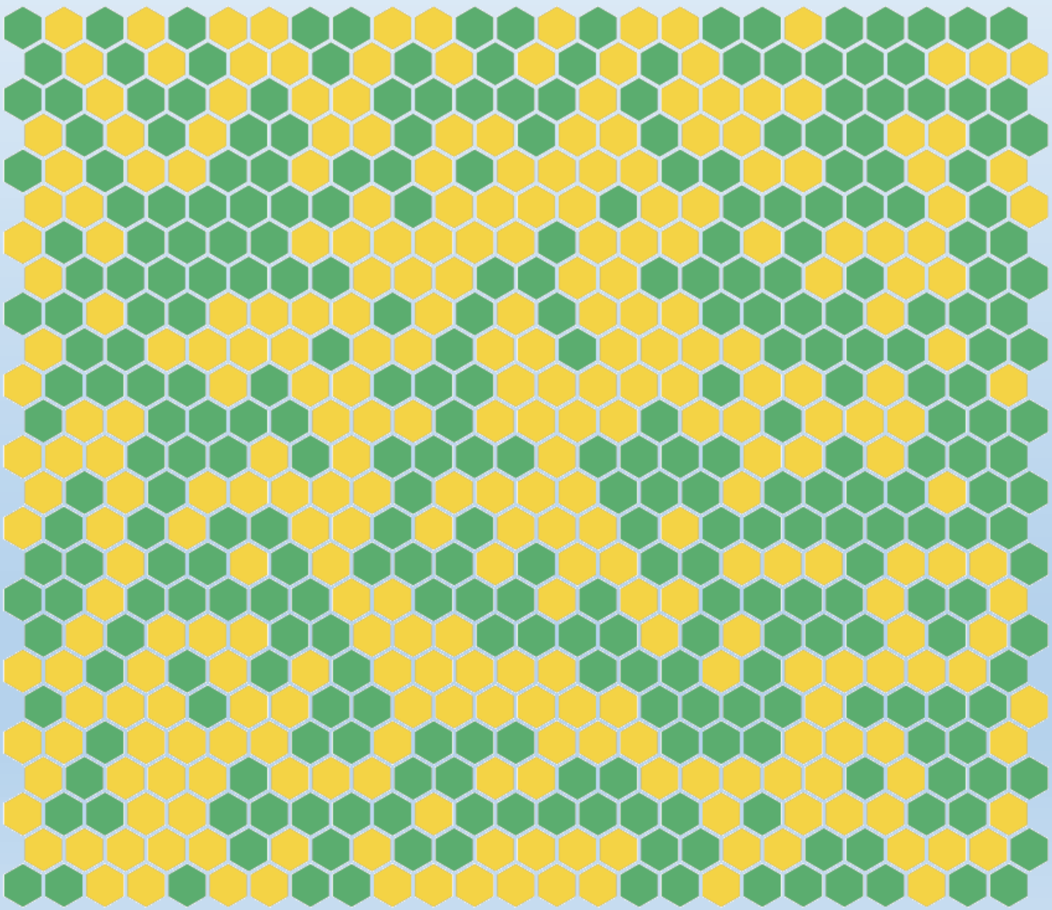
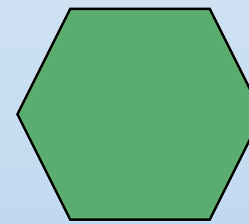
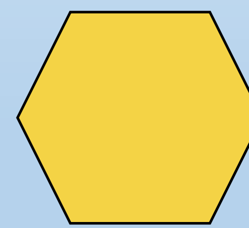


$1 =$ 

$-1 =$ 

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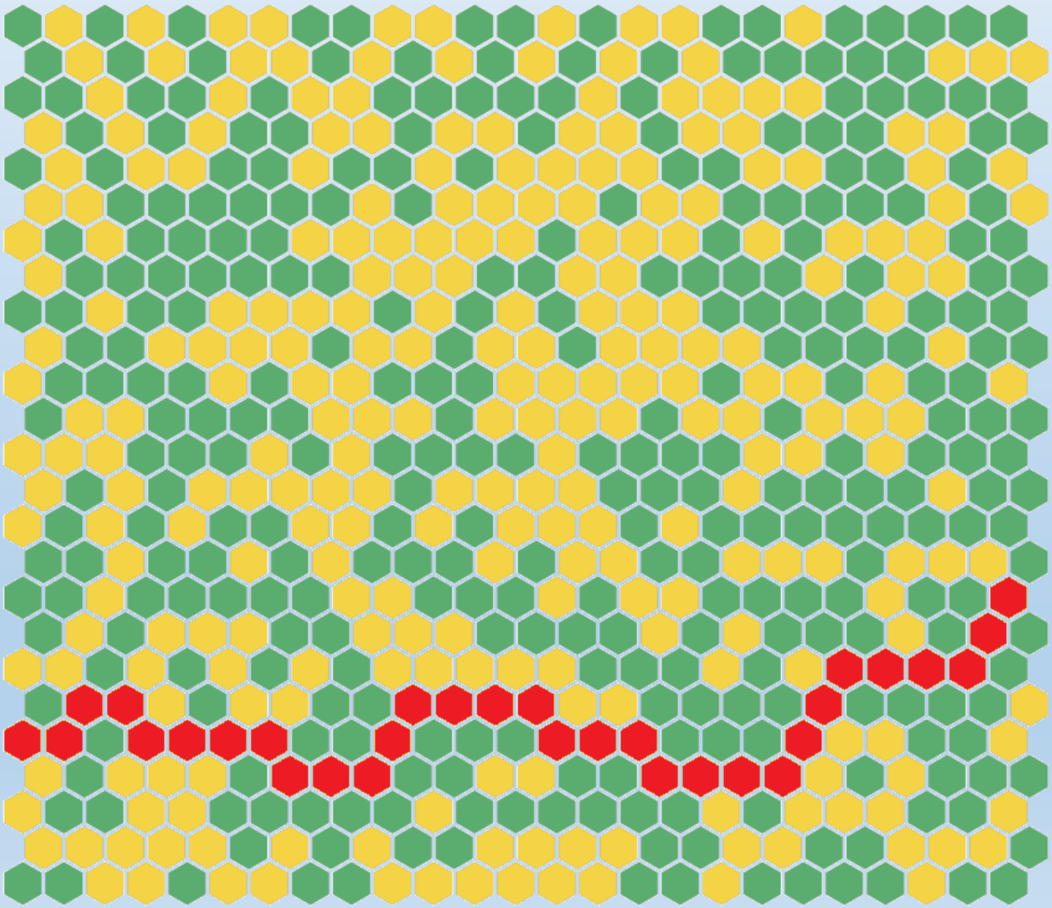
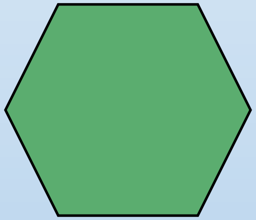
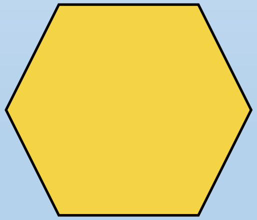
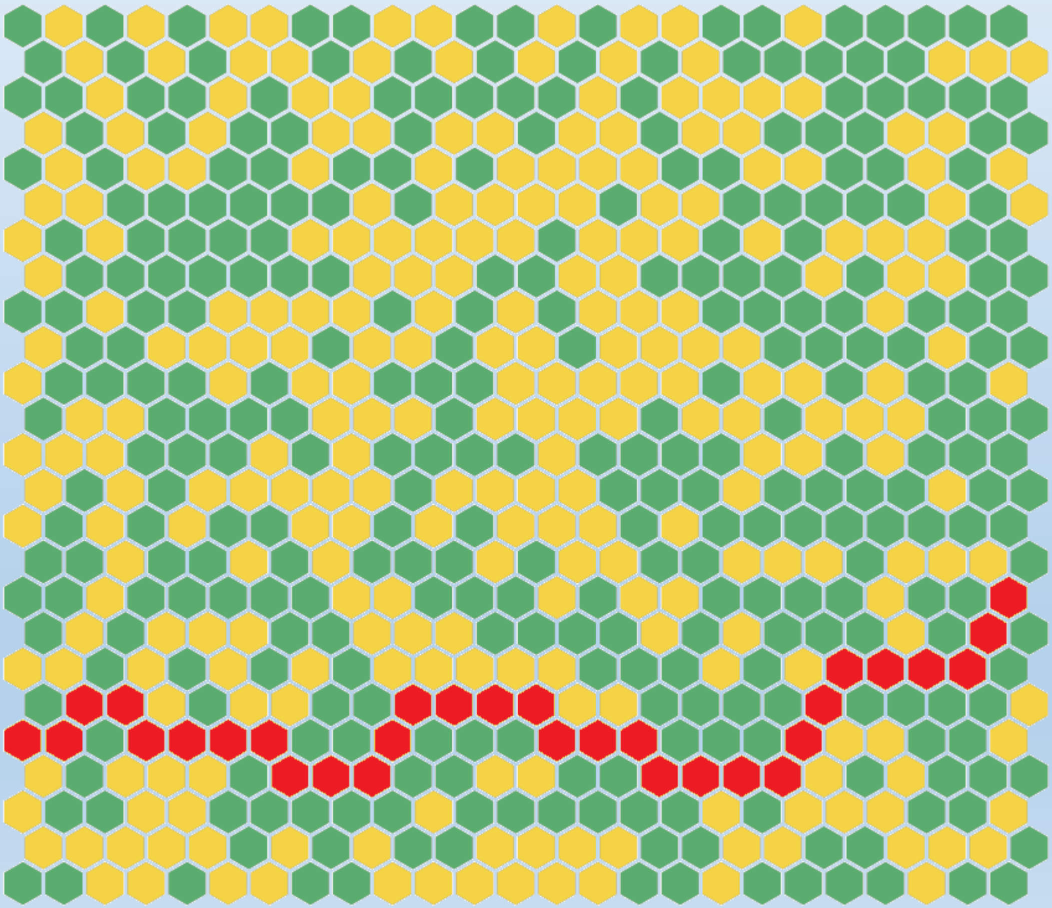


$1 =$ 

$-1 =$ 

Percolation

$$f(x) = \begin{cases} 1 & \text{if green} \updownarrow \text{crossing} \\ -1 & \text{if yellow} \leftrightarrow \text{crossing} \end{cases}$$



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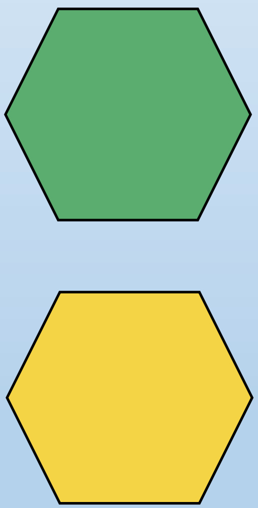
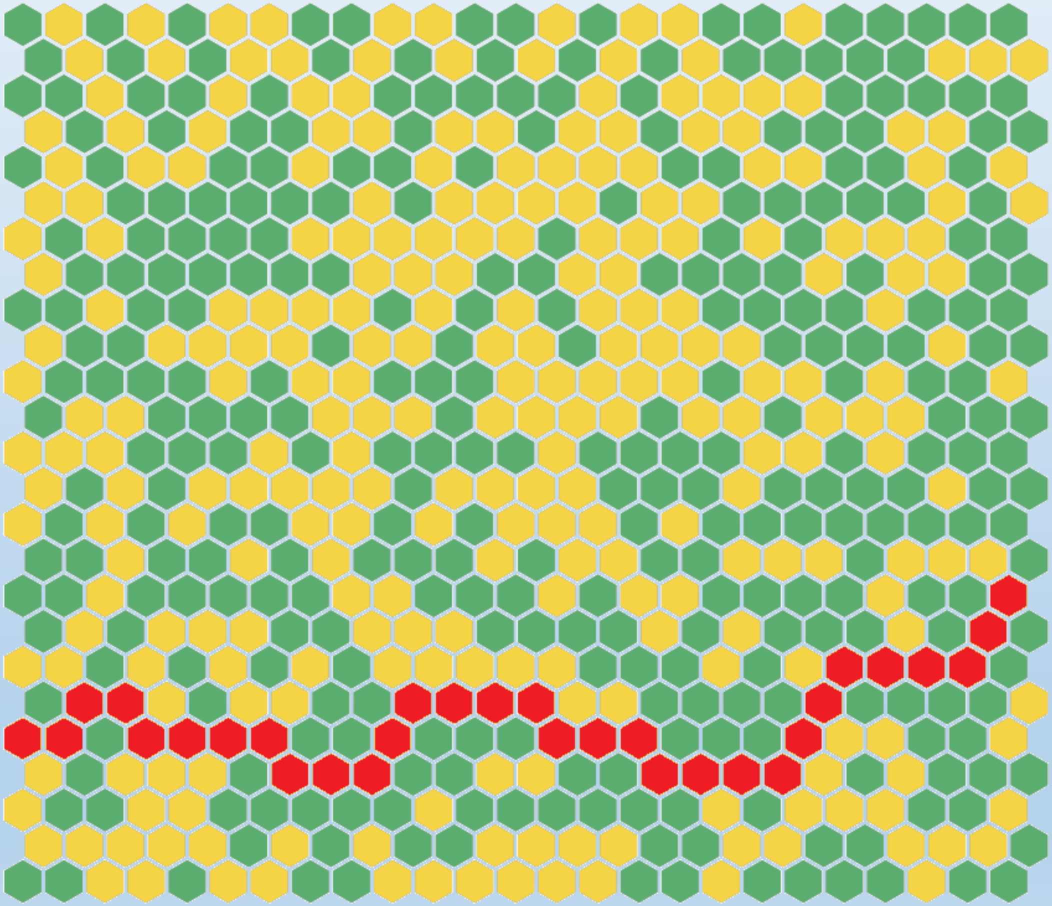


Noise sensitivity



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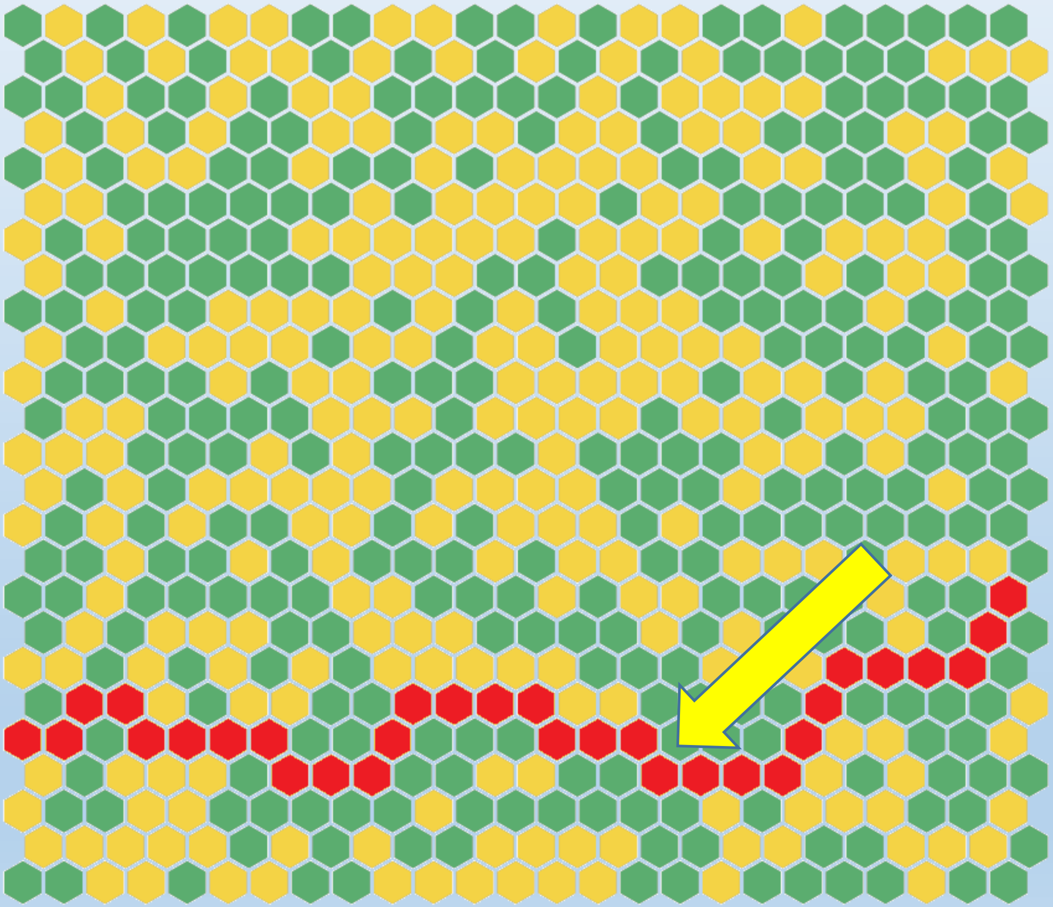

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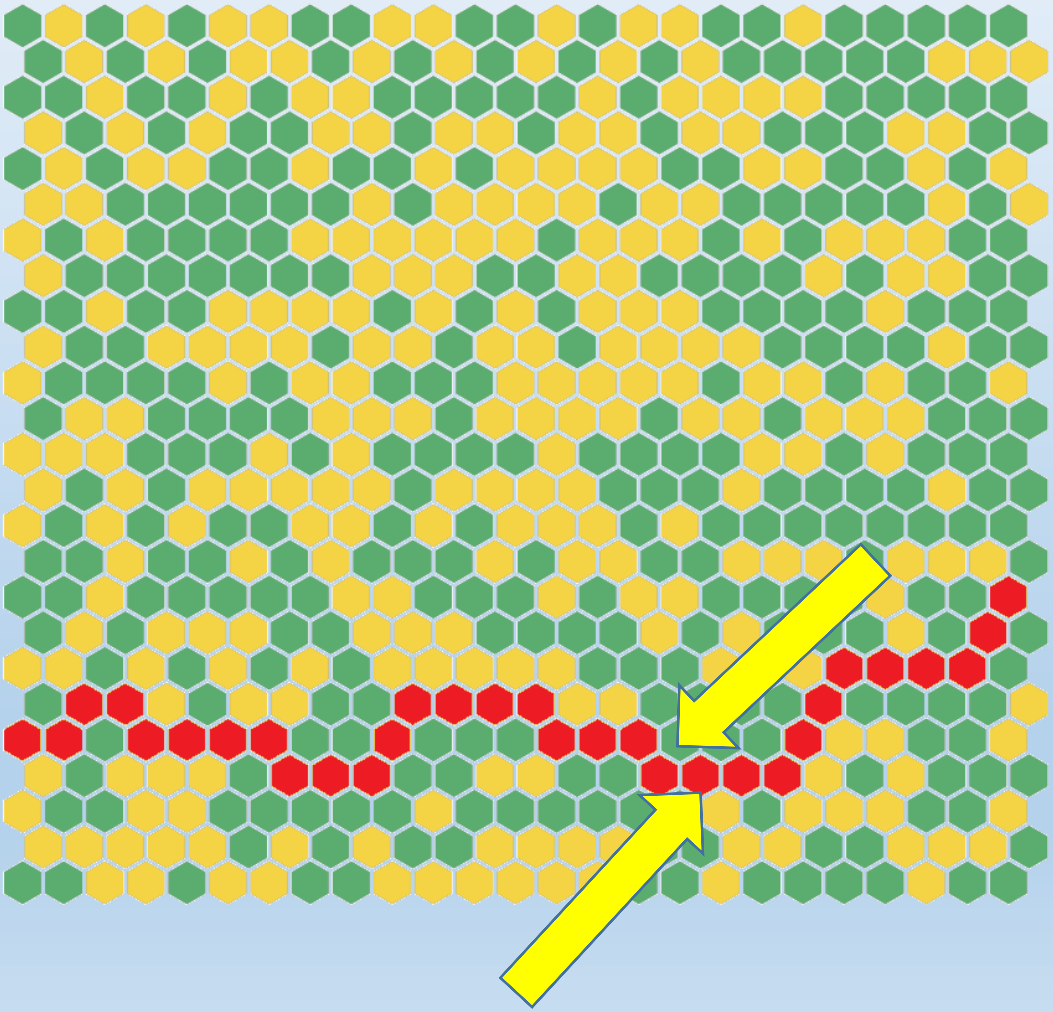

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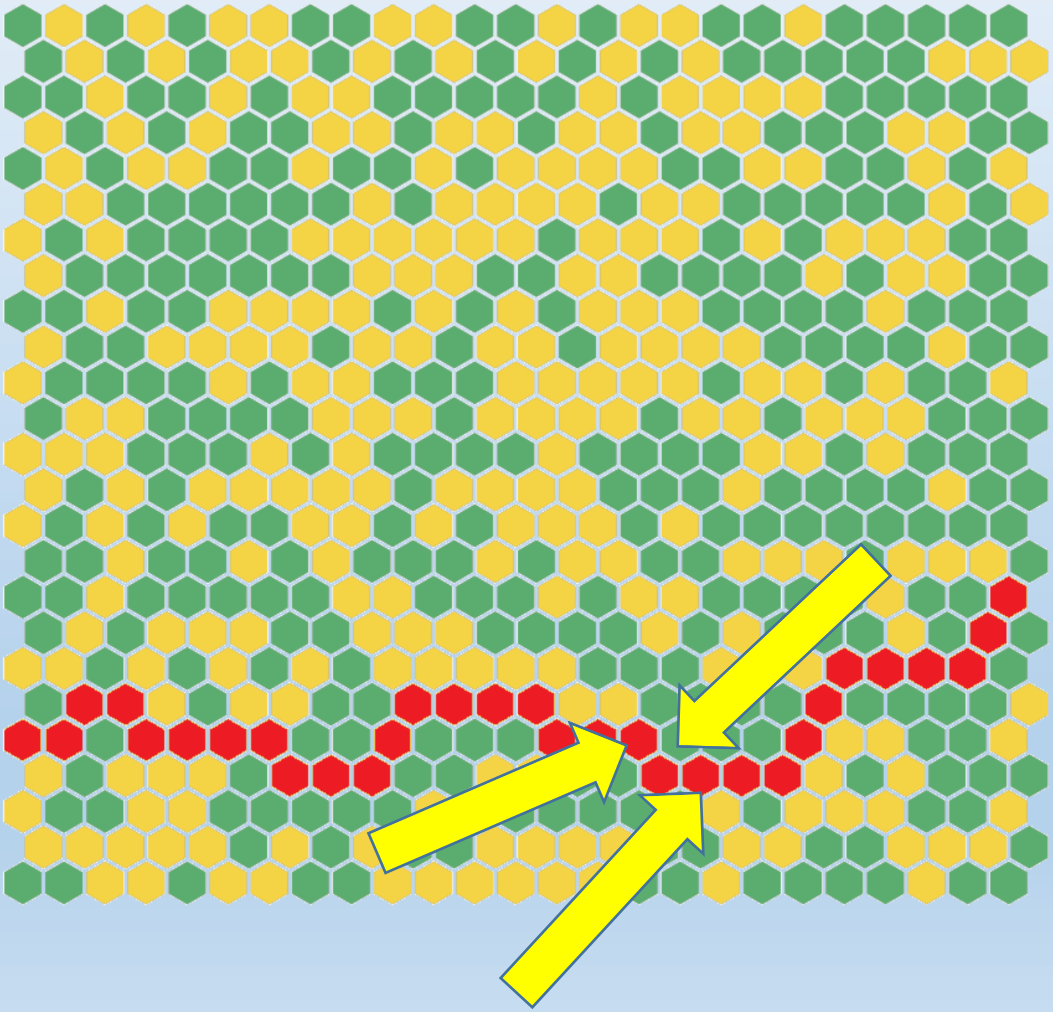

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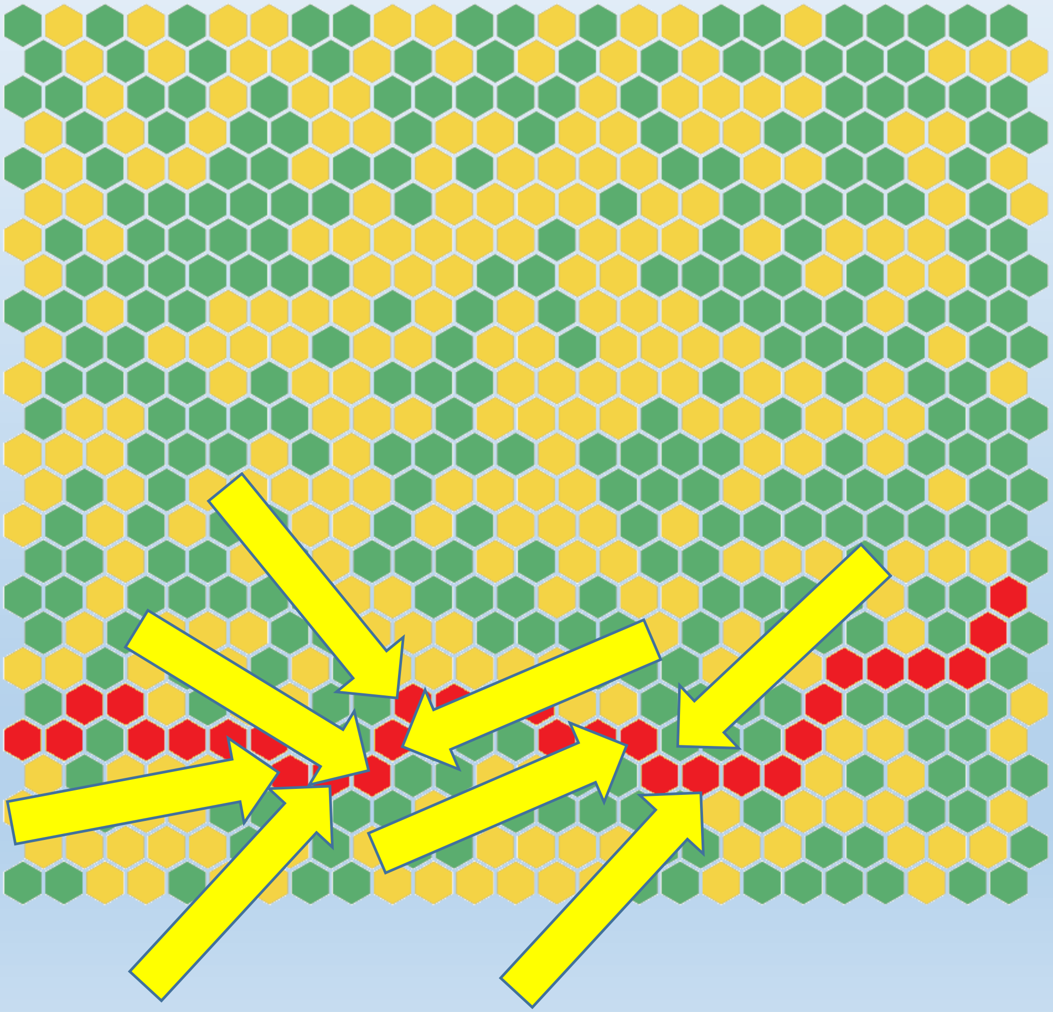

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Pick $\varepsilon > 0$, and flip each bit with probability ε.

Did the function's value change?

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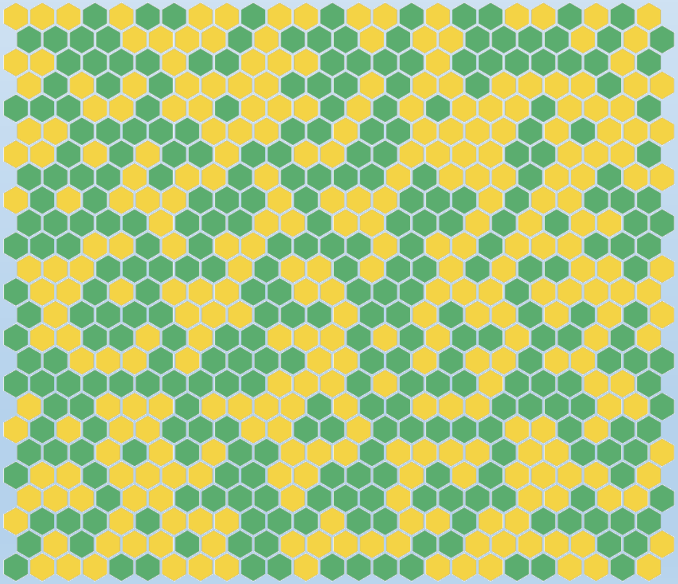

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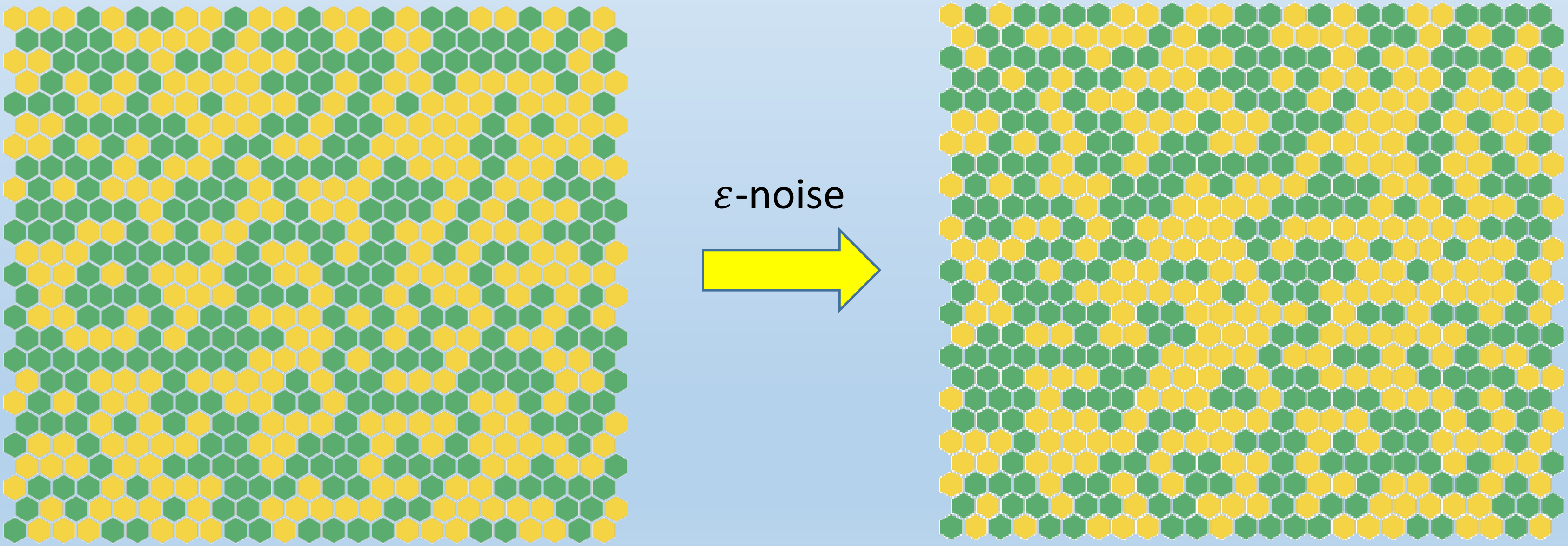

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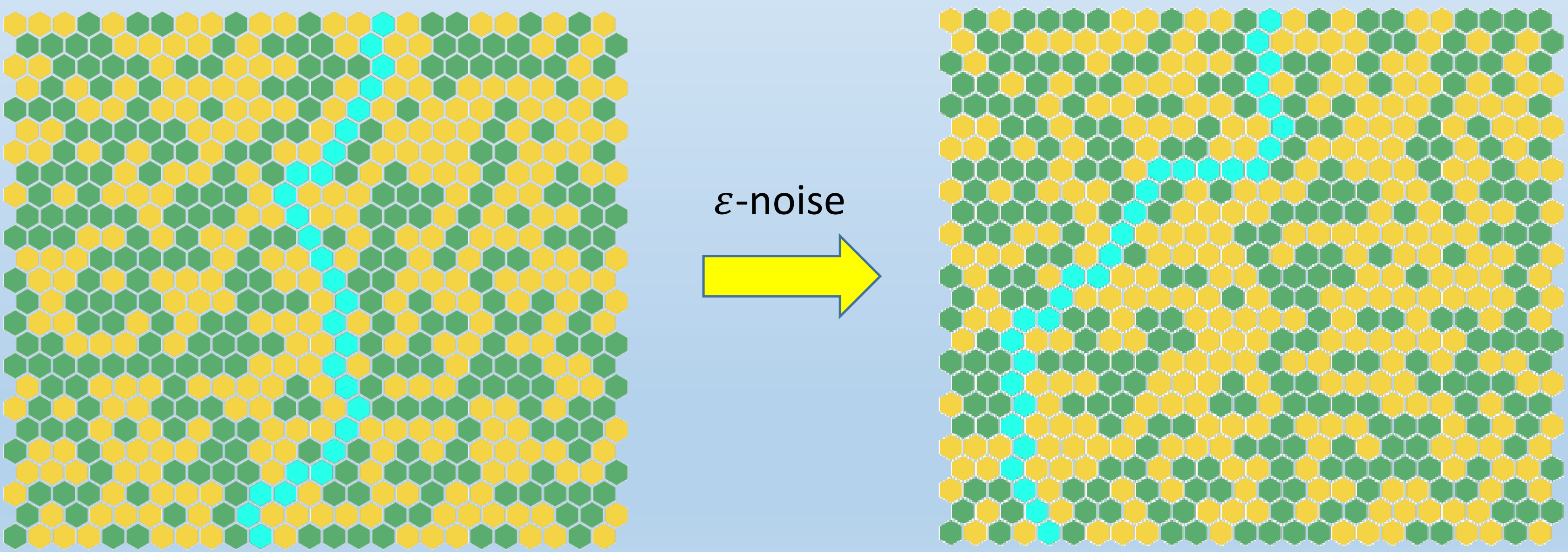

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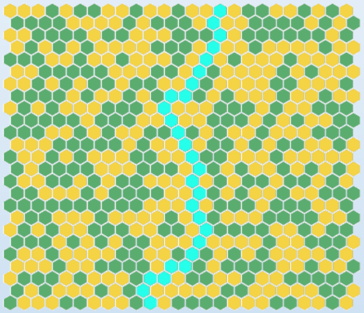

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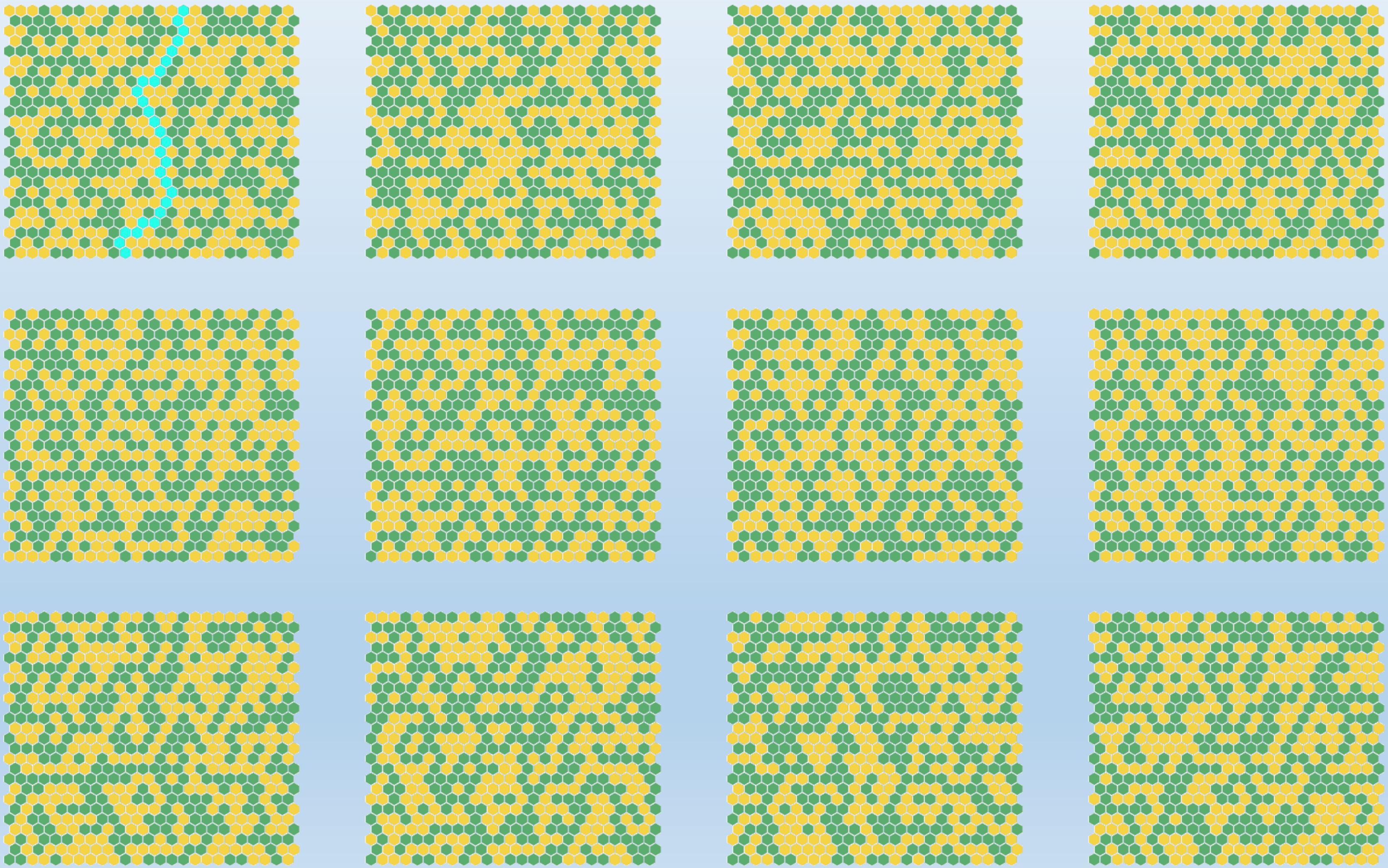

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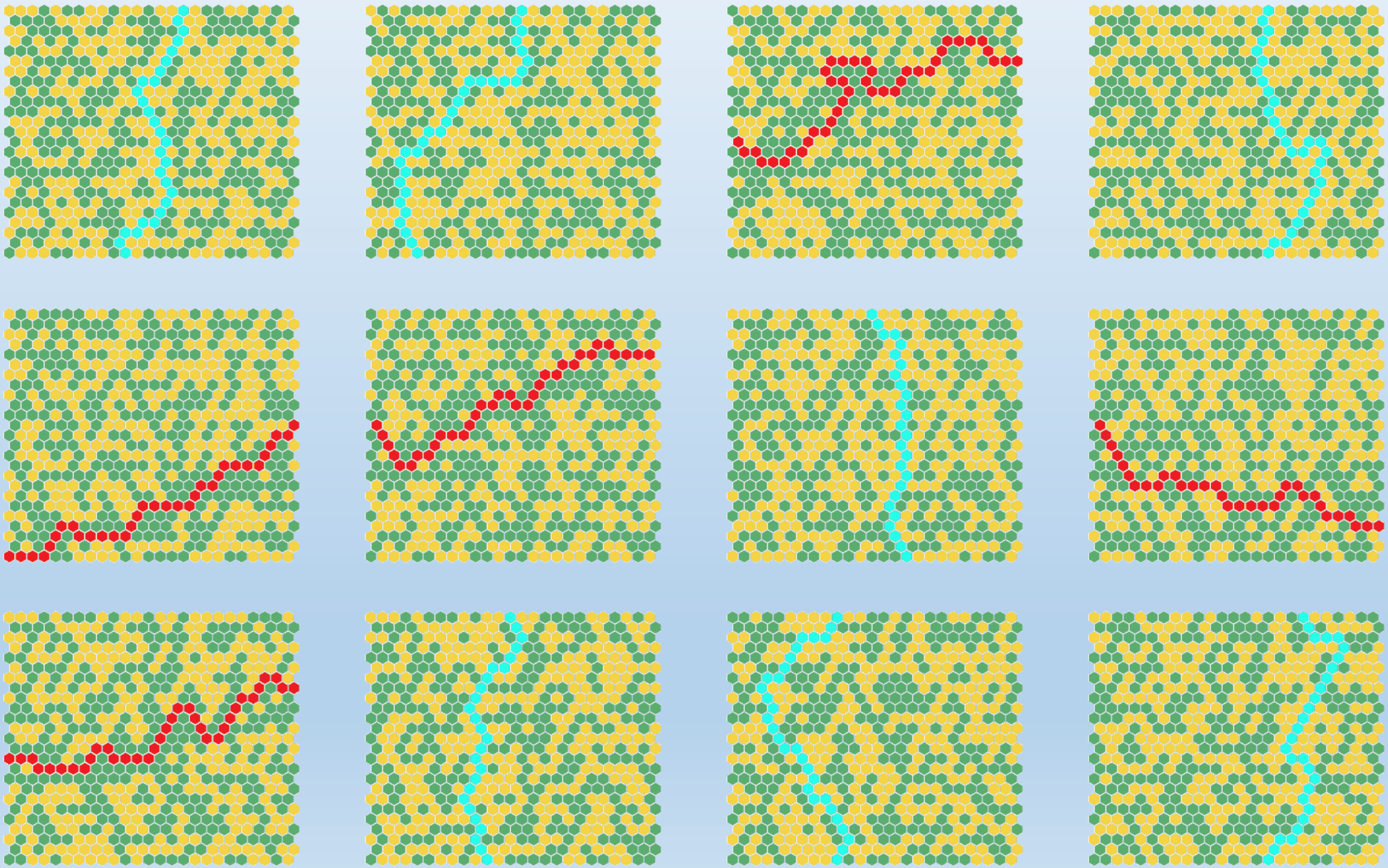

Noise sensitivity

Definition: A sequence $f_n: \{-1,1\}^n \to \{-1,1\}$ of balanced Boolean functions is called "*noise sensitive*" if for all $\varepsilon > 0$,

$$\lim_{n\to\infty} \mathbb{E}[f_n(x) f_n(y)] = 0,$$

where x is random and y is an ε-noising of x.

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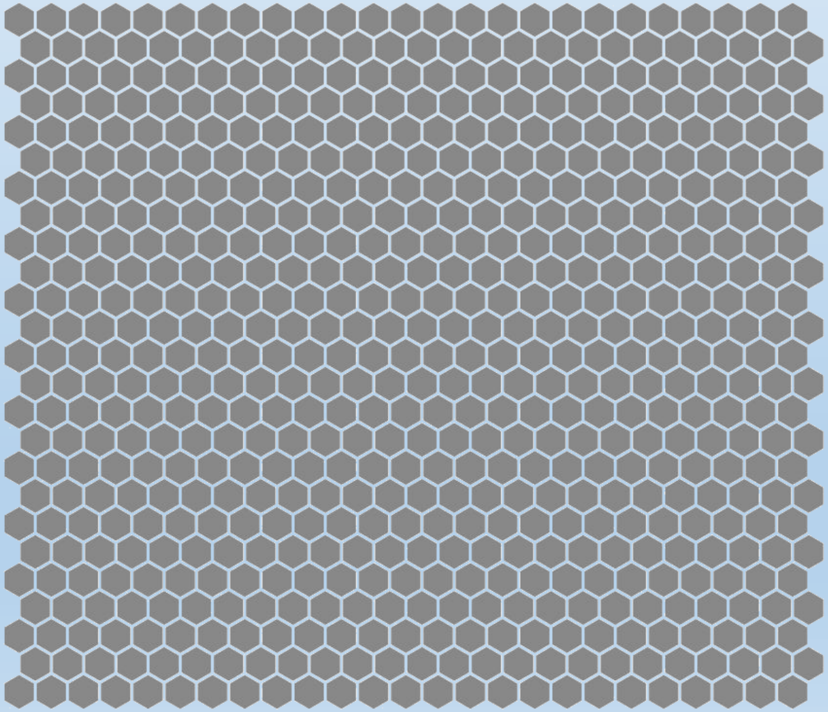
Is percolation crossing noise sensitive?

If so, how fast can ε go to 0 with n?

Decision trees

x is uniform random, but hidden from you.

Reveal hexagons one by one, until $f(x)$ is found.



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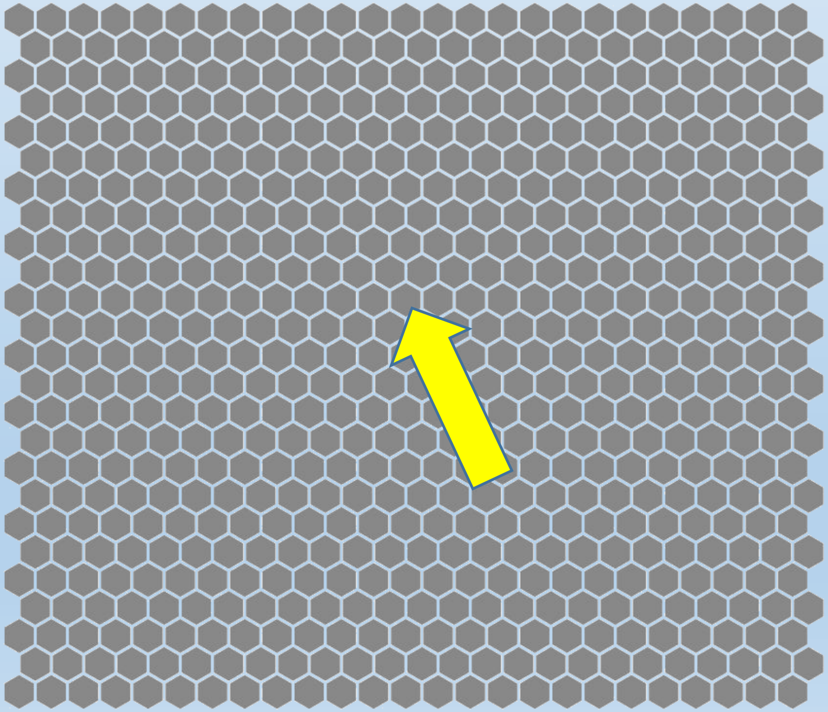

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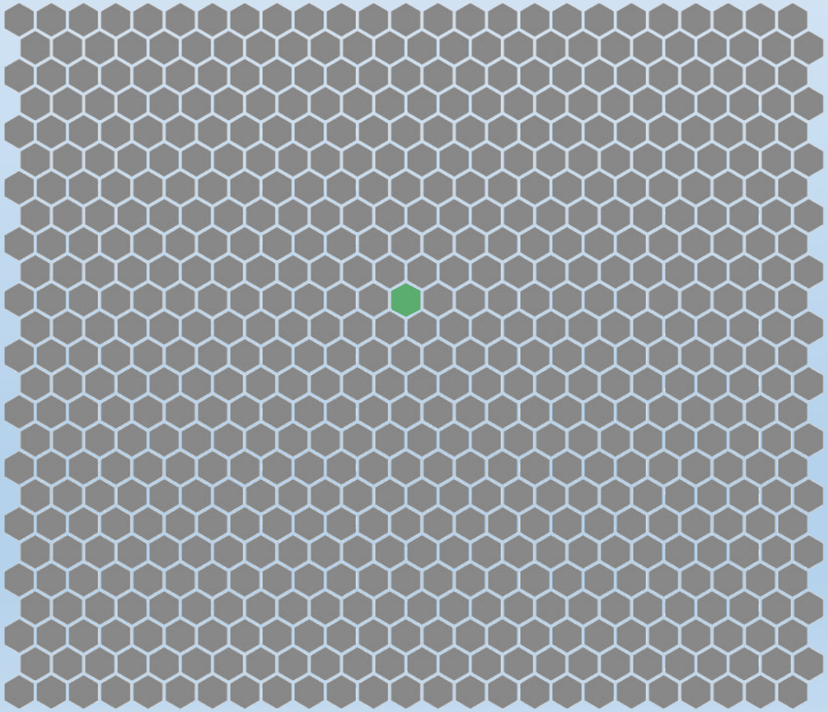

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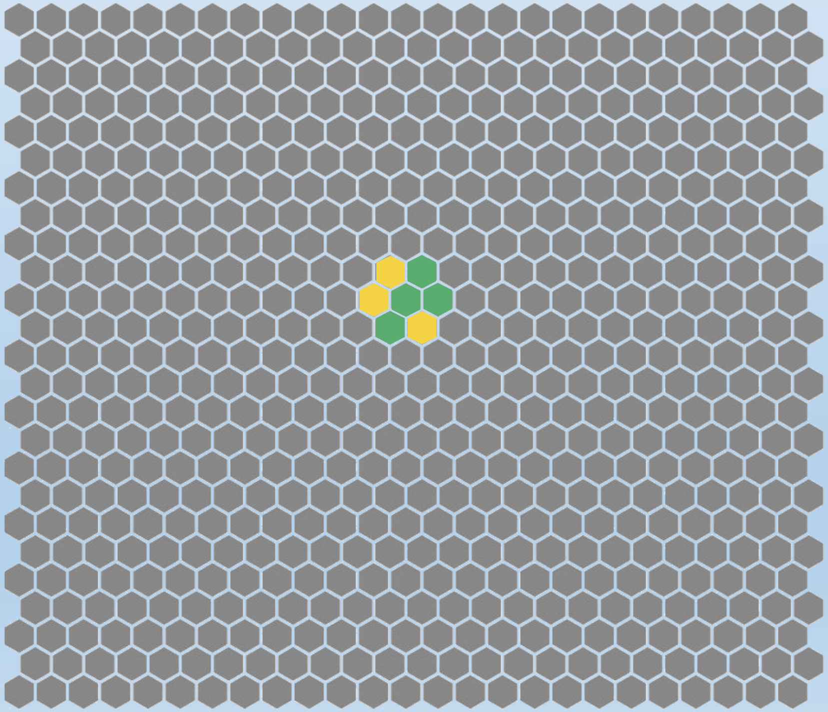

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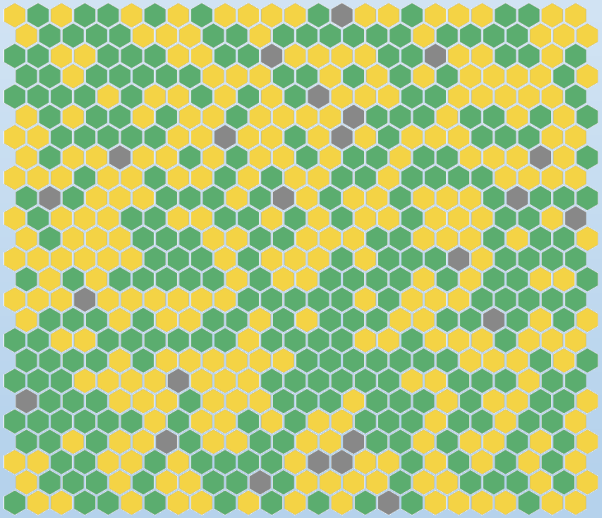

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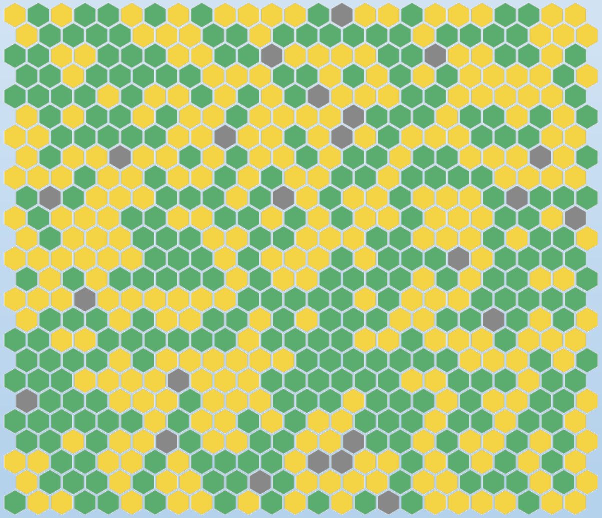

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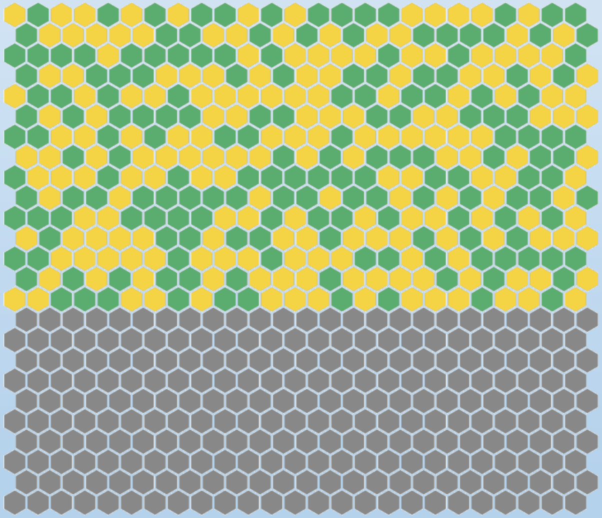

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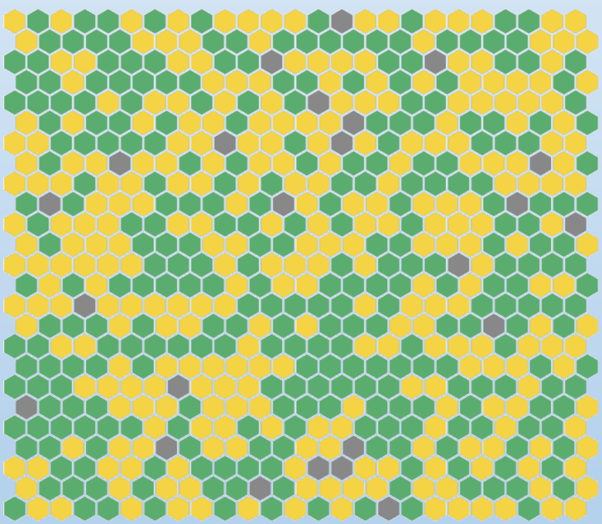

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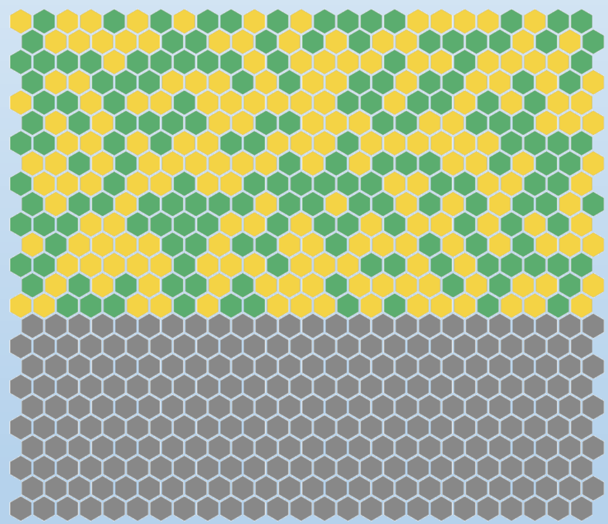

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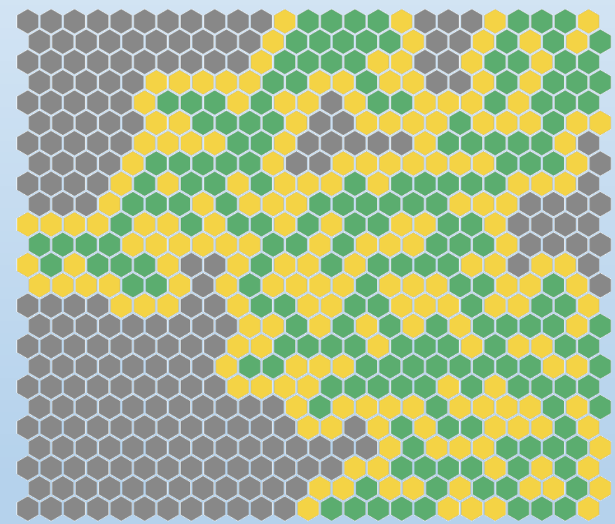

Random floodfill

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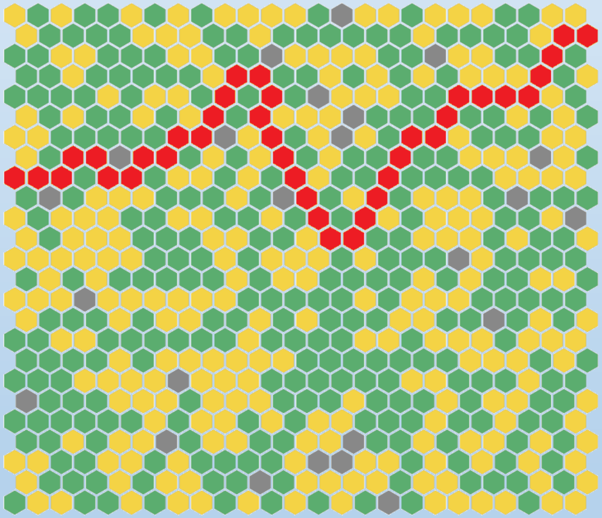

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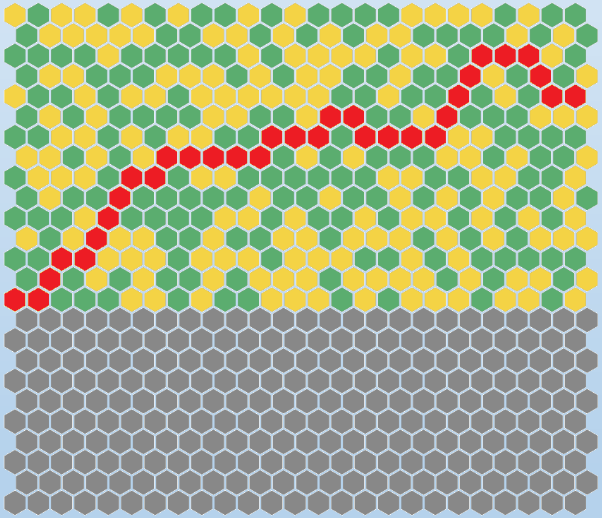

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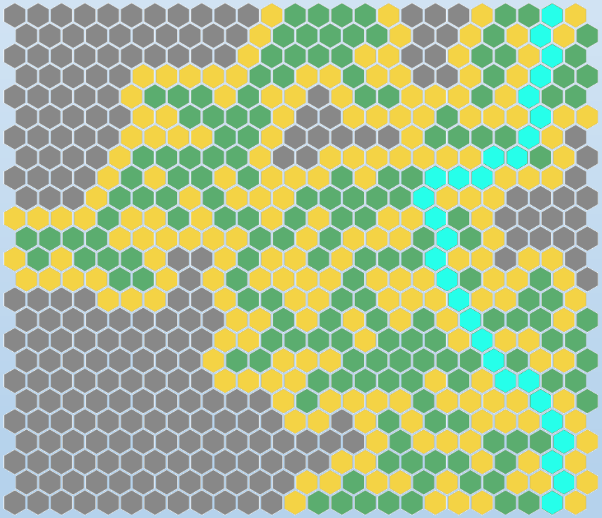

Random floodfill

The Schramm-Steif Theorem

Let $f_n : \{-1,1\}^n \to \{-1,1\}$ be a sequence of Booelan functions.

Let T_n be a bit-reveal algorithm for f_n, and

$$\delta(n) := \max_i \delta_i = \max_i \mathbb{P}[T_n \text{ reveals bit } i]\,.$$

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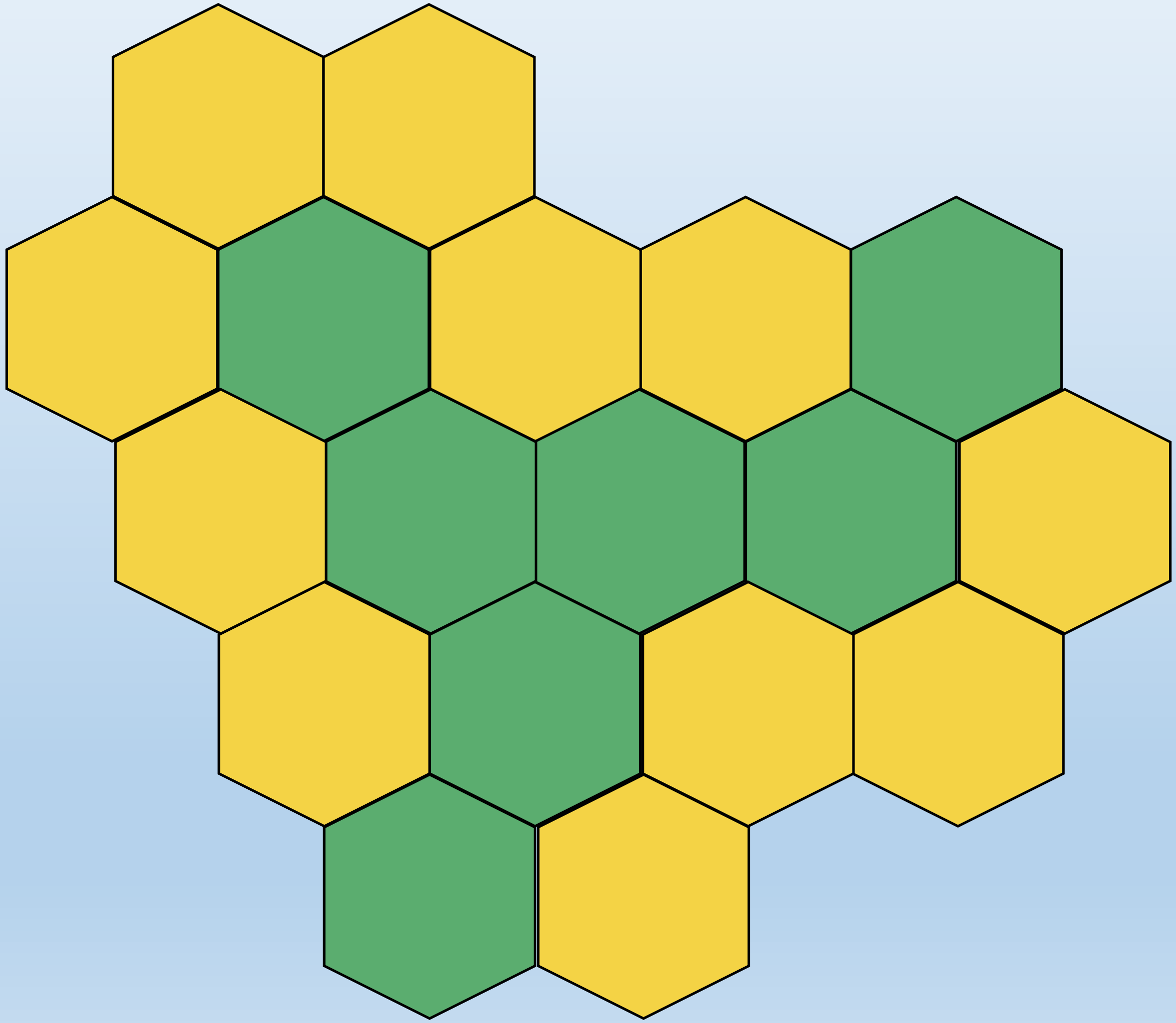
Theorem: If $\delta \to 0$, then f_n is noise sensitive.

The faster $\delta \to 0$, the more noise sensitive it is!

The interface algorithm



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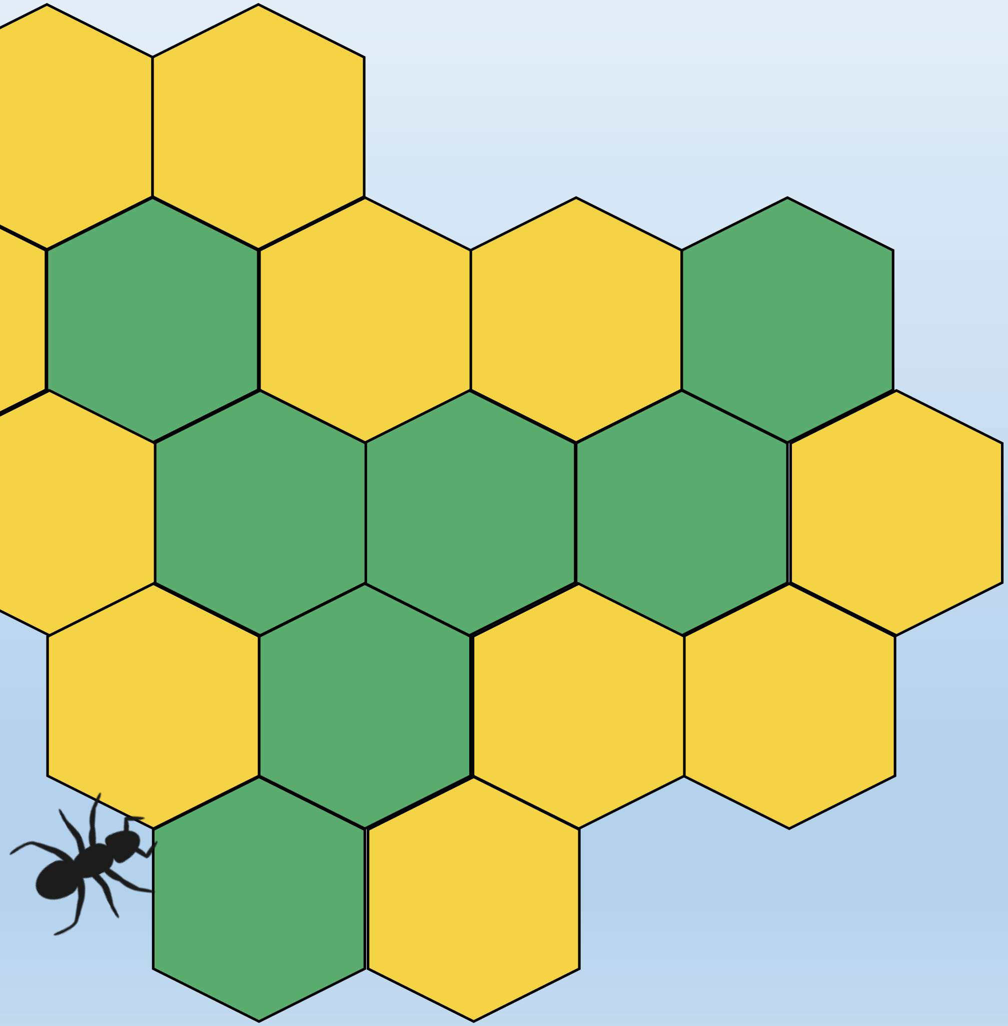

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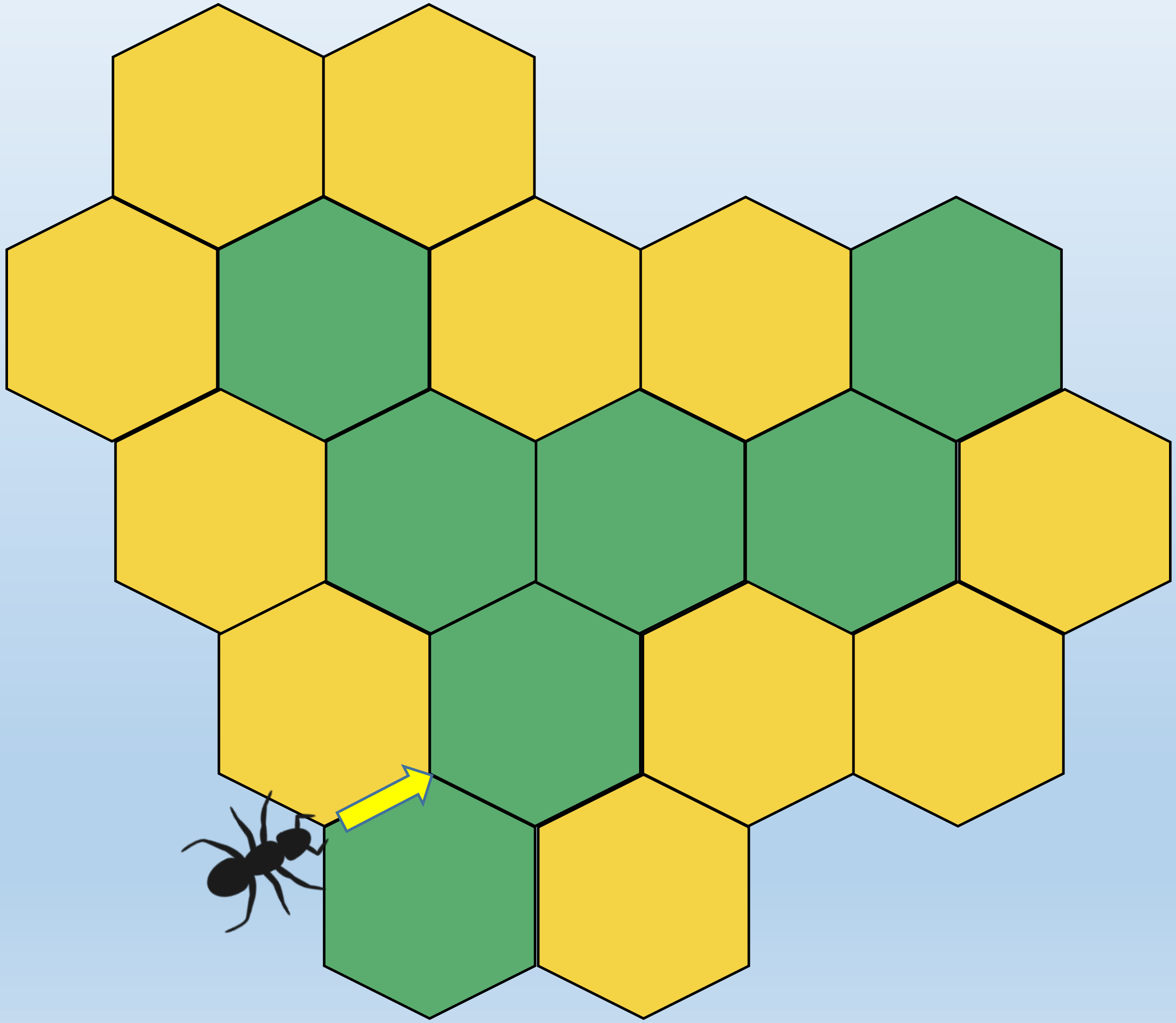

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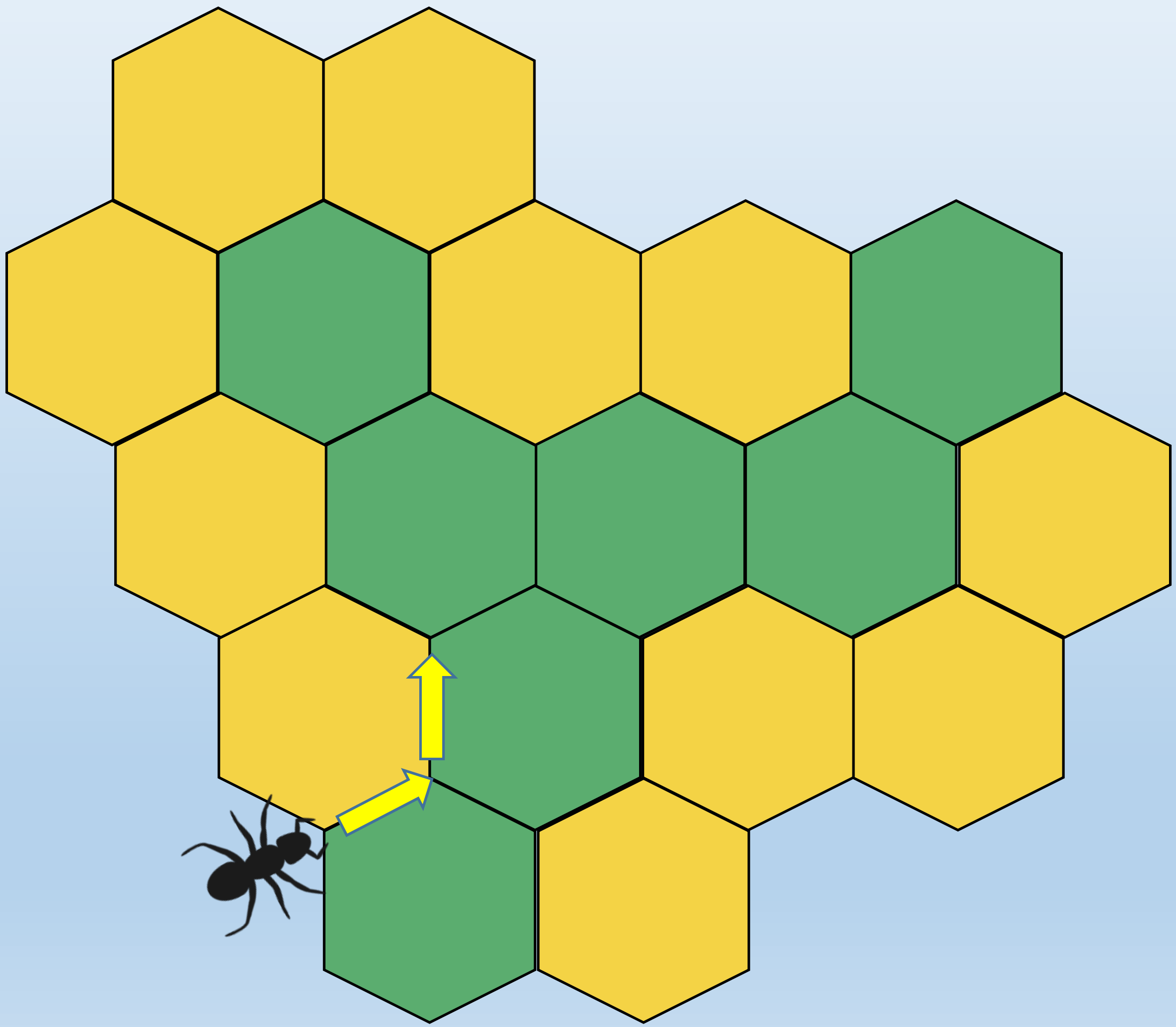

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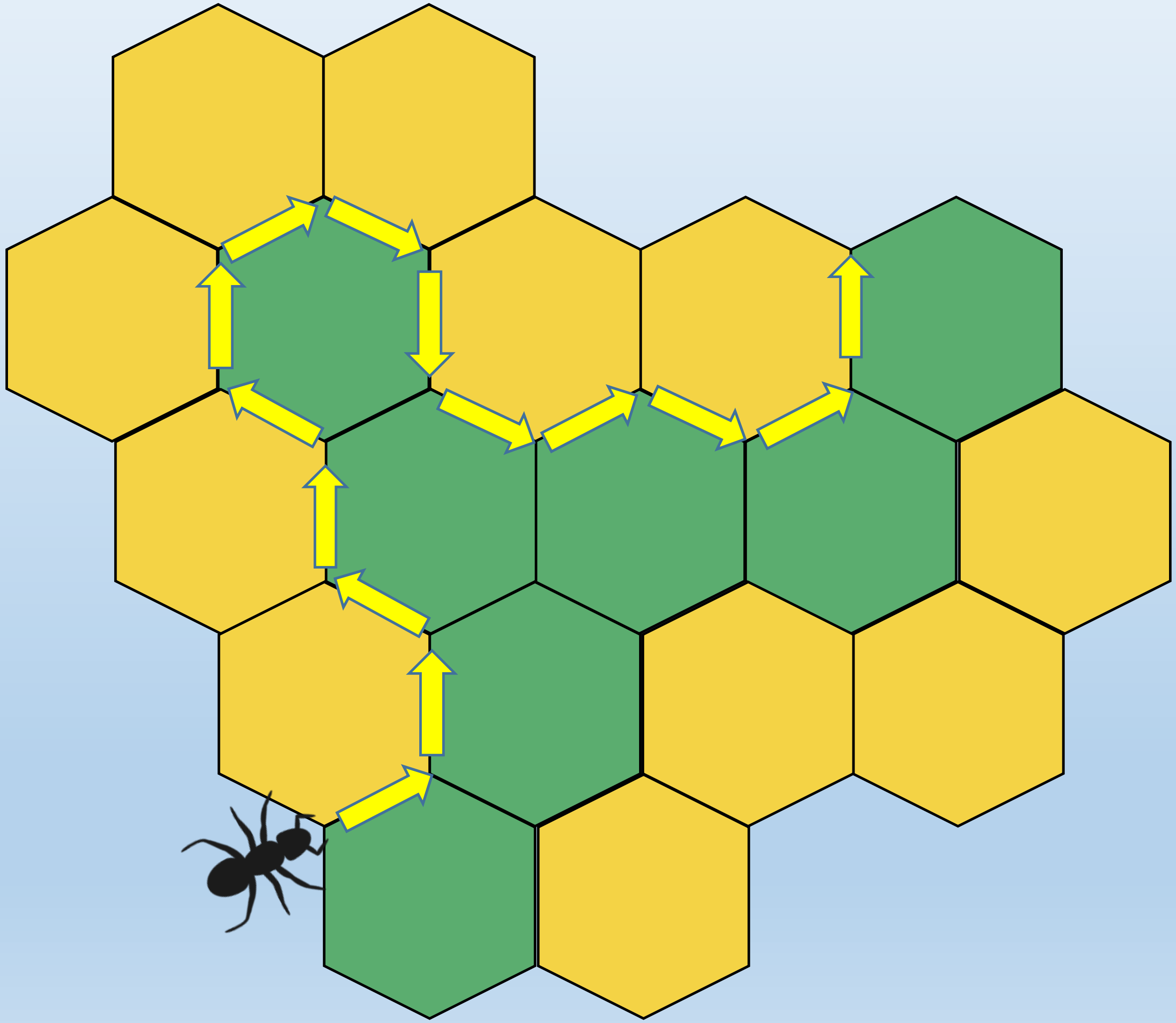

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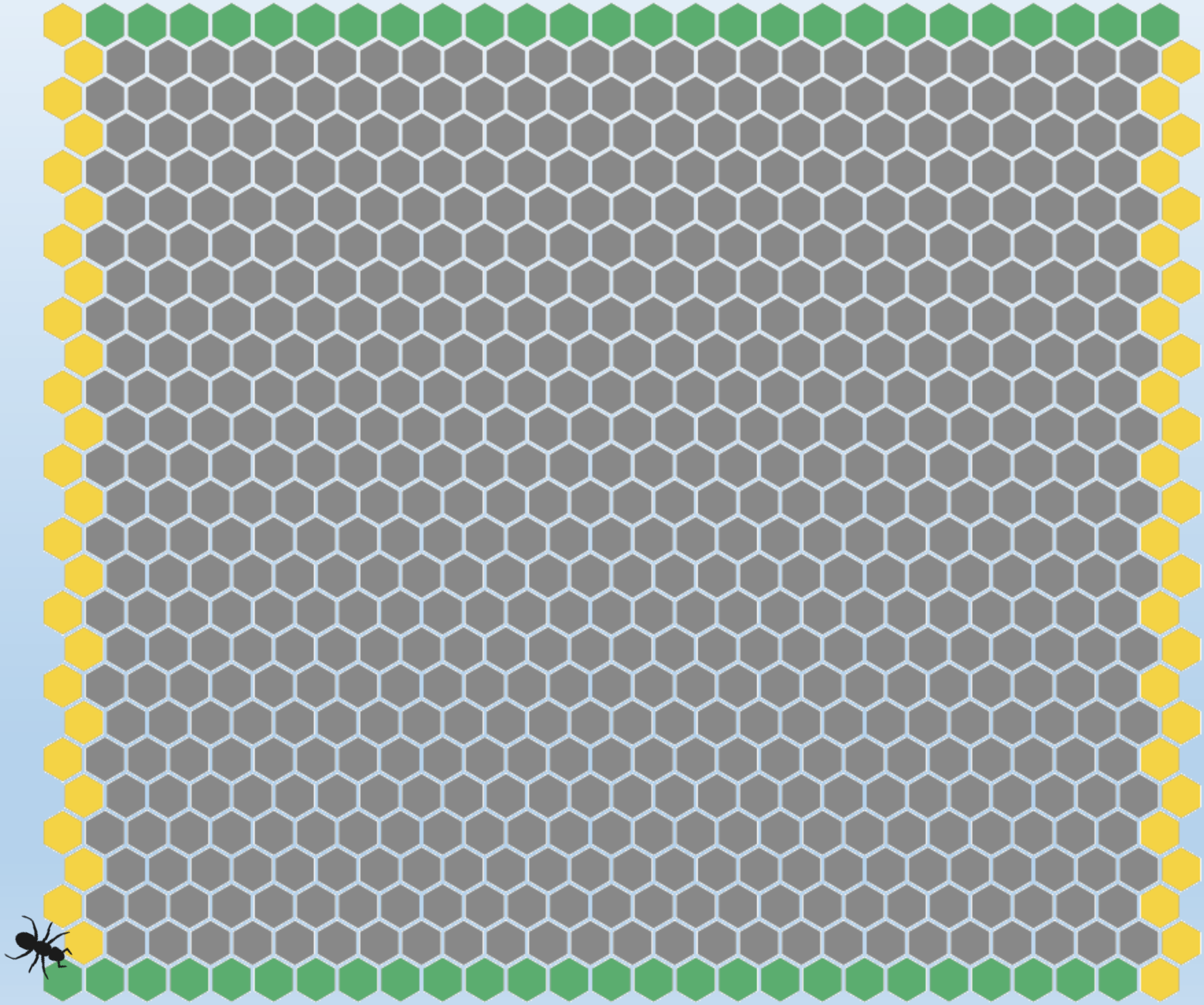

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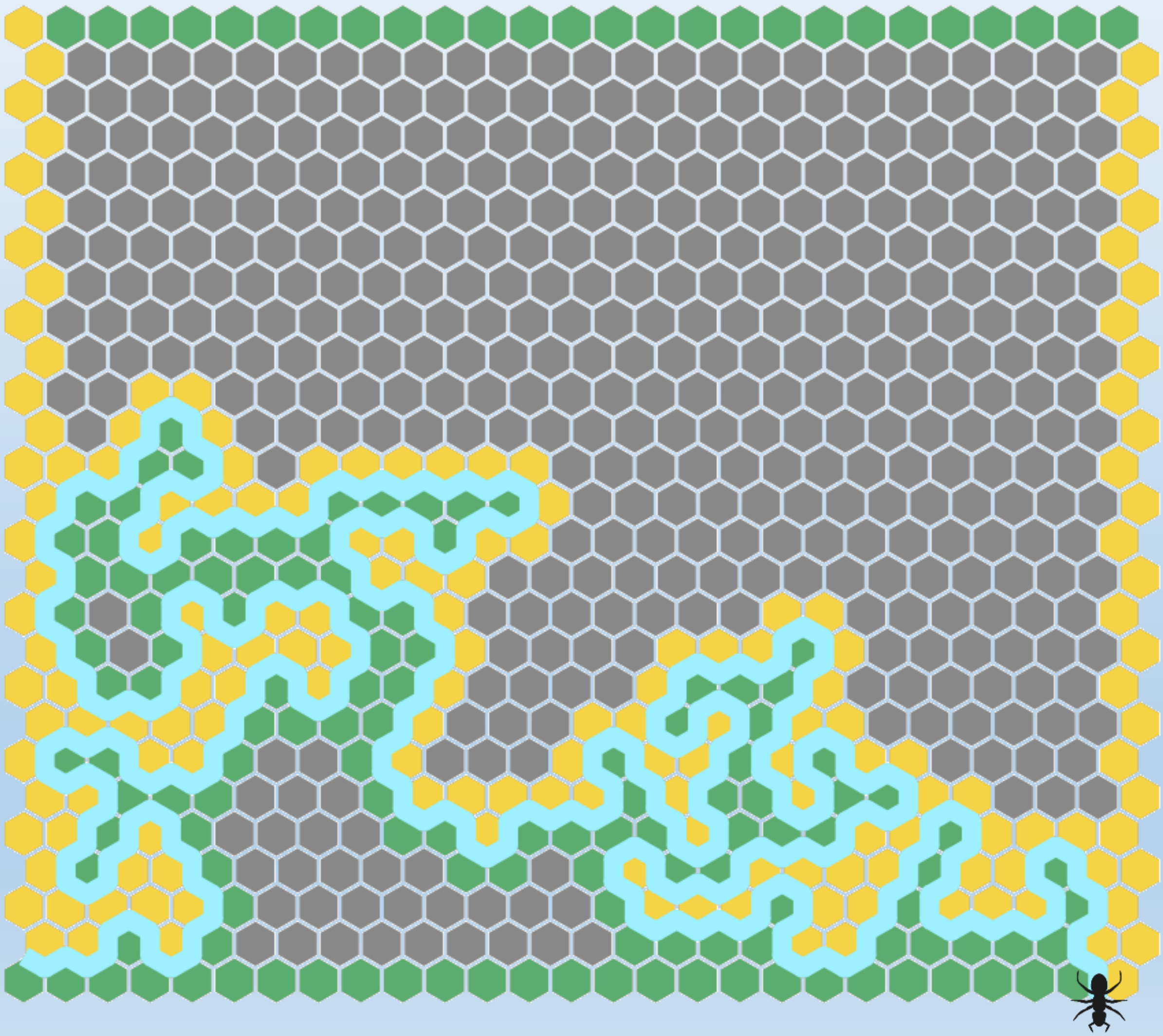

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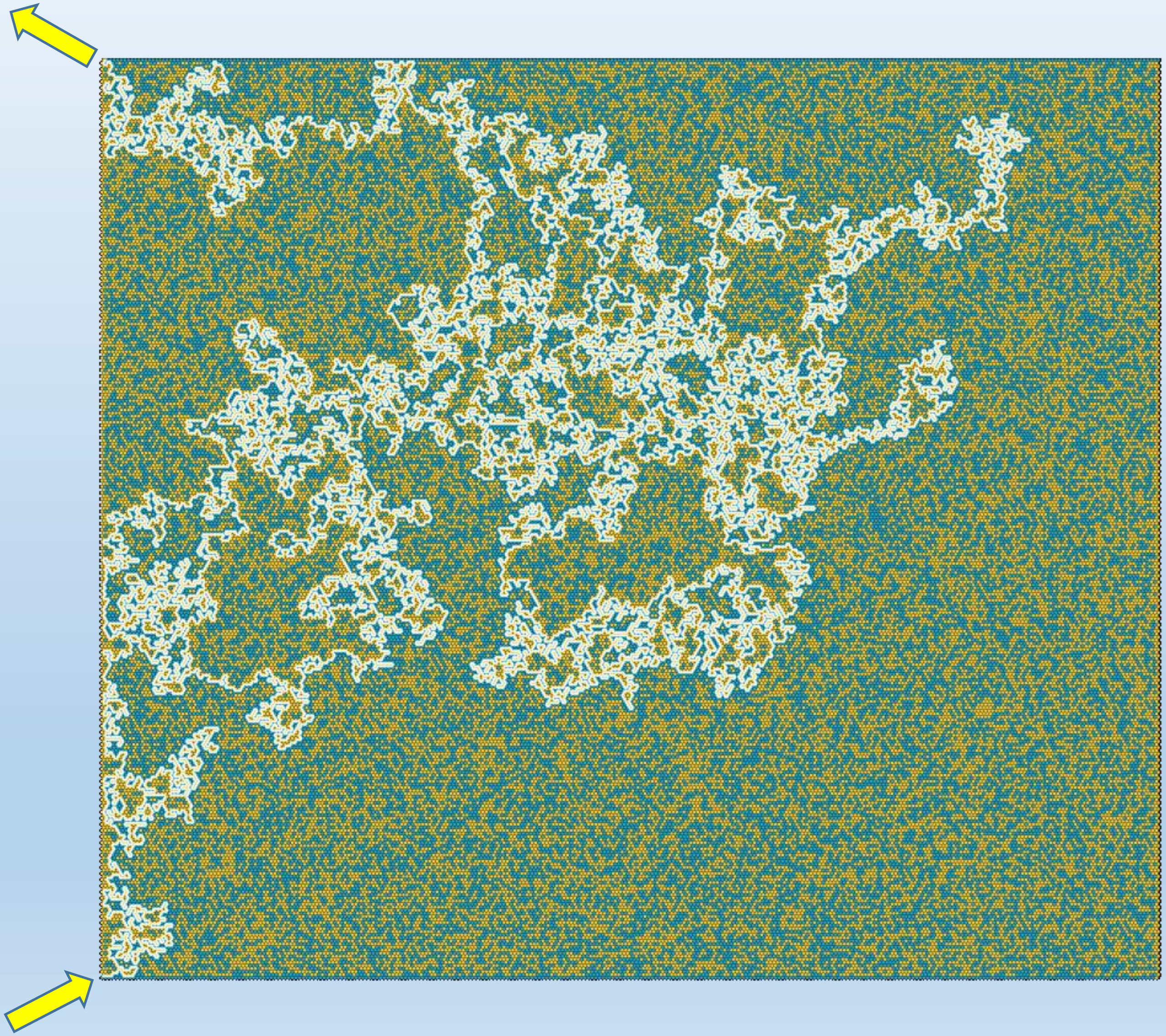

Fractional query algorithms

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View 1:

True x	1	-1	1	-1	-1	1	1	-1

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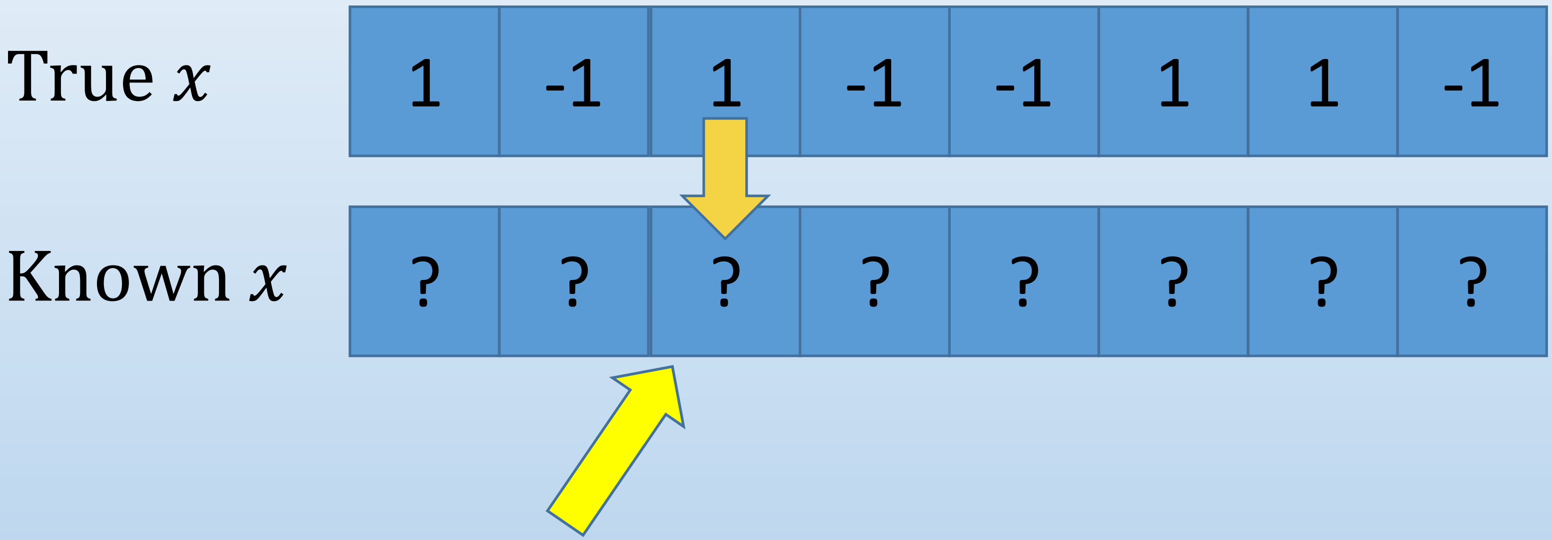

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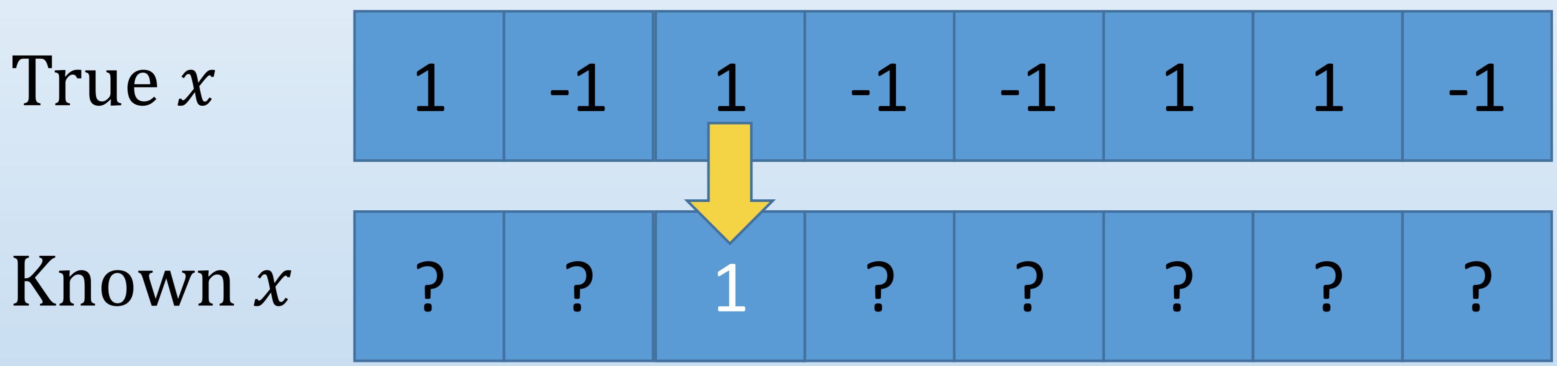

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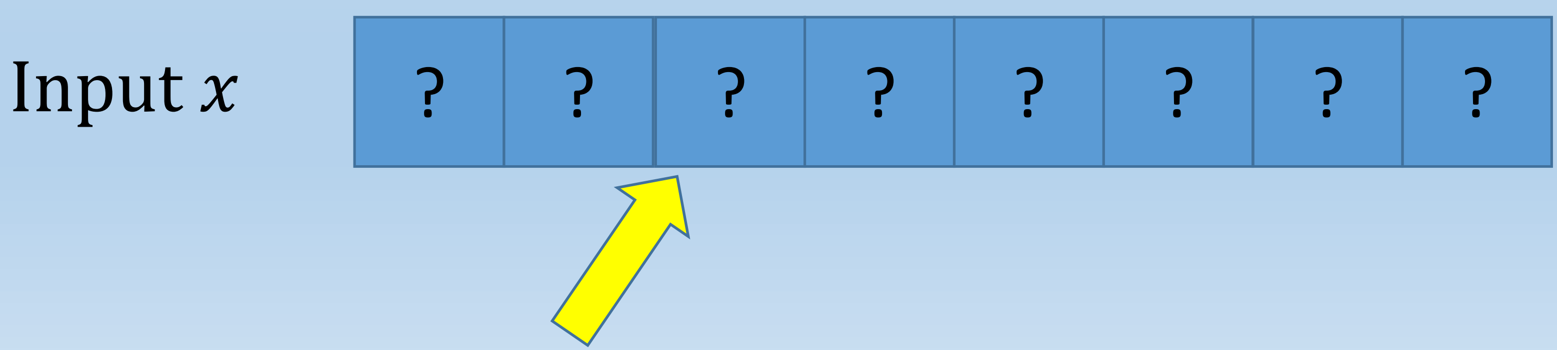

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$$x_i = \begin{cases} 1 & \text{w.p. } 1/2 \\ -1 & \text{w.p. } 1/2 \end{cases}$$

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	0	0	0	0	0	0	0	0

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$x(1)$	0	0	0	1	0	0	0	0	

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$x(0)$	0	0	0	0	0	0	0	0
$x(1)$	0	0	0	1	0	0	0	0
$x(2)$	0	0	0	1	0	-1	0	0



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$x(0)$	0	0	0	0	0	0	0	0	
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$x(2)$	0	0	0	1	0	-1	0	0	
$\vdots$				$\vdots$					
$x(\tau)$	1	0	0	1	0	-1	-1	1	

Fractional query algorithms

Input $x(t)$ is a stochastic process!

$x(0)$	0	0	0	0	0	0	0	0	
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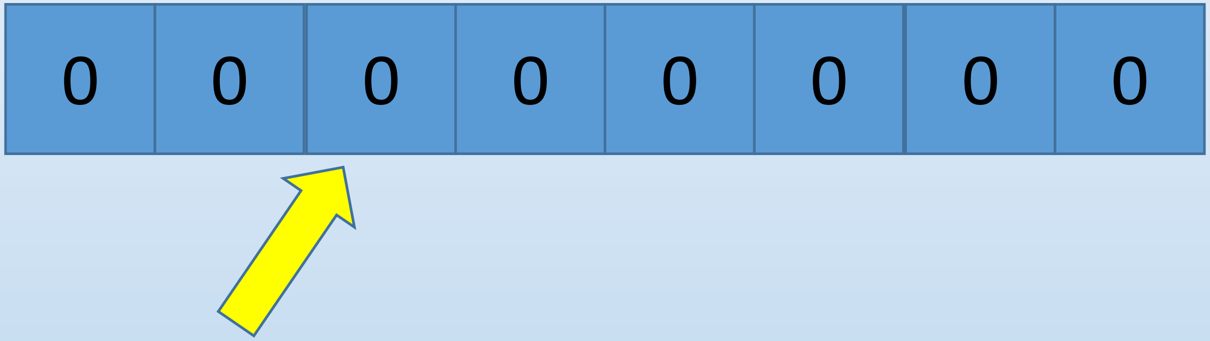

Fractional query algorithms

Let $\varepsilon > 0$.

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	0	0	0	0	0	0	0	0

$$x_i = x_i + \begin{cases} \varepsilon & \text{w.p. } 1/2 \\ -\varepsilon & \text{w.p. } 1/2 \end{cases}$$

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Can only do this if $x_i(t) \in (-1,1)$.

Computing with fractional inputs

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$$f(x) = \sum_{S \subseteq [n]} \hat{f}(S) \prod_{i \in S} x_i$$

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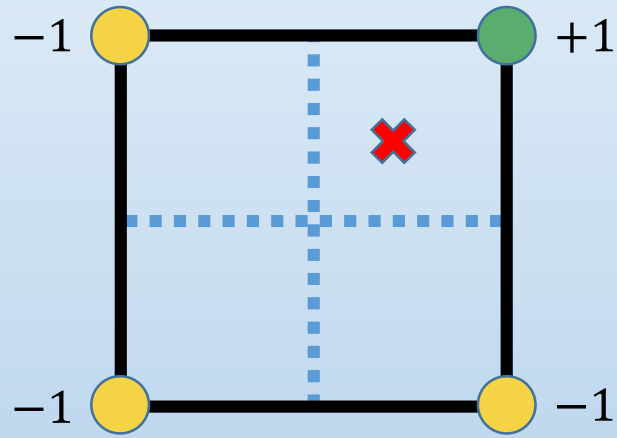

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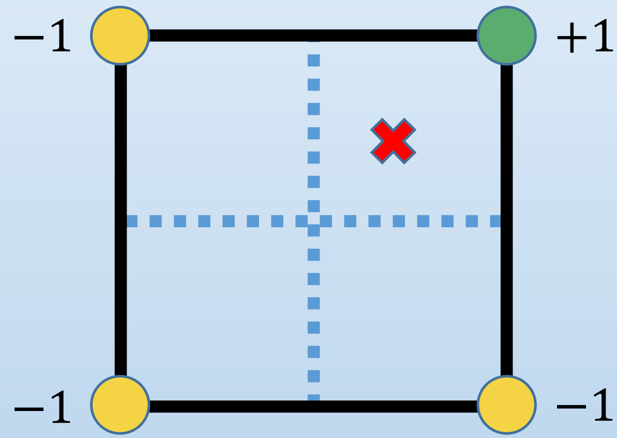

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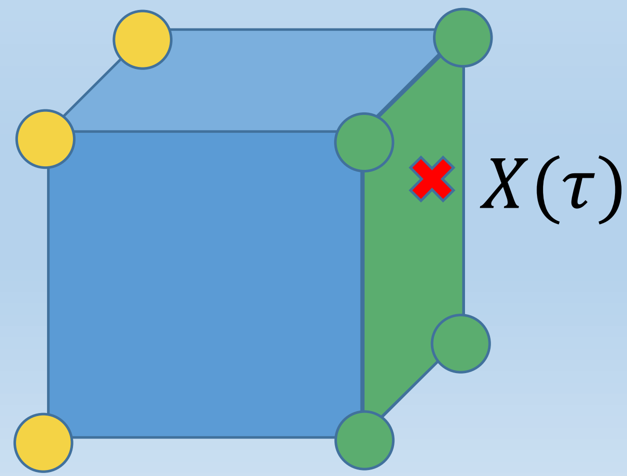

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Where is the unknown input?

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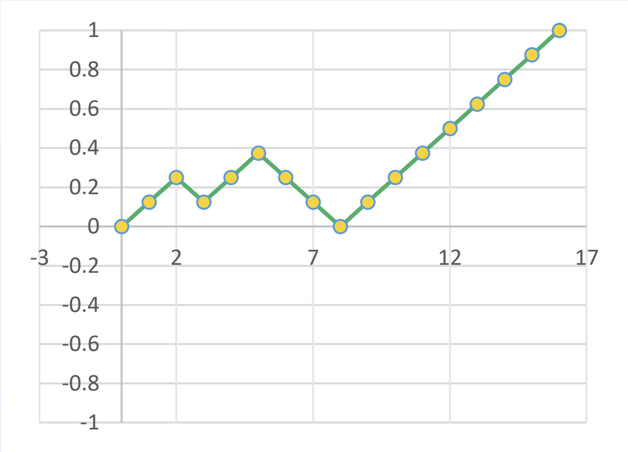
Individual bits perform random walks



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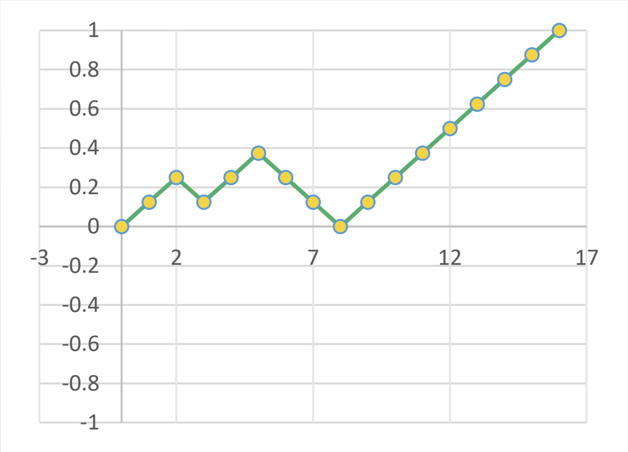

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This is the "final input".

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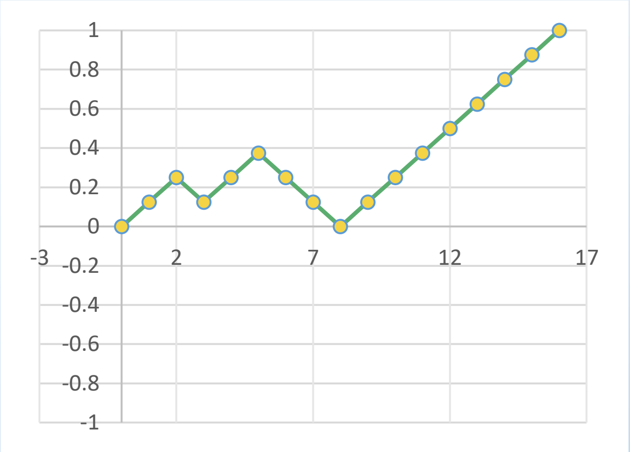

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The current value gives a hint to the future:

$$\mathbb{P}[\, X_i(\infty) = 1 \mid X_i(t) \,] = \frac{1 + X_i(t)}{2}$$

Comparing with classical algorithms

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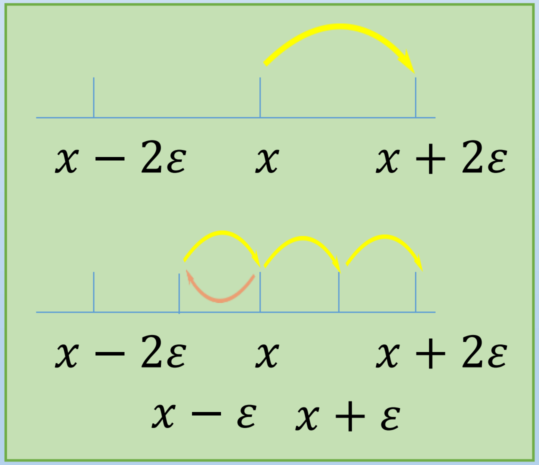

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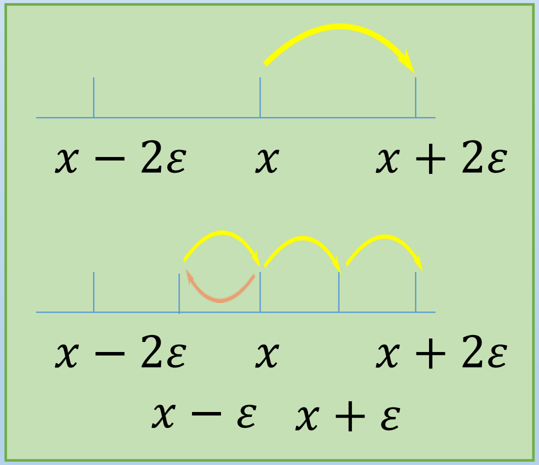

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Why this cost?

Fact: $\mathbb{E}[X_i(\tau)^2] = \mathbb{E}[X_i]_\tau = \varepsilon^2 \mathbb{E}[\#\text{times } i \text{ was chosen}]$

The Schramm-Steif Theorem

Let $f_n: \{-1,1\}^n \to \{-1,1\}$ be a sequence of Booelan functions.

Let T_n be a bit-reveal algorithm for f_n, and

$$\delta(n) := \max_i \delta_i = \max_i \mathbb{P}[T_n \text{ reveals bit } i]\,.$$

Theorem: If $\delta \to 0$, then f_n is noise sensitive.

The faster $\delta \to 0$, the more noise sensitive it is!

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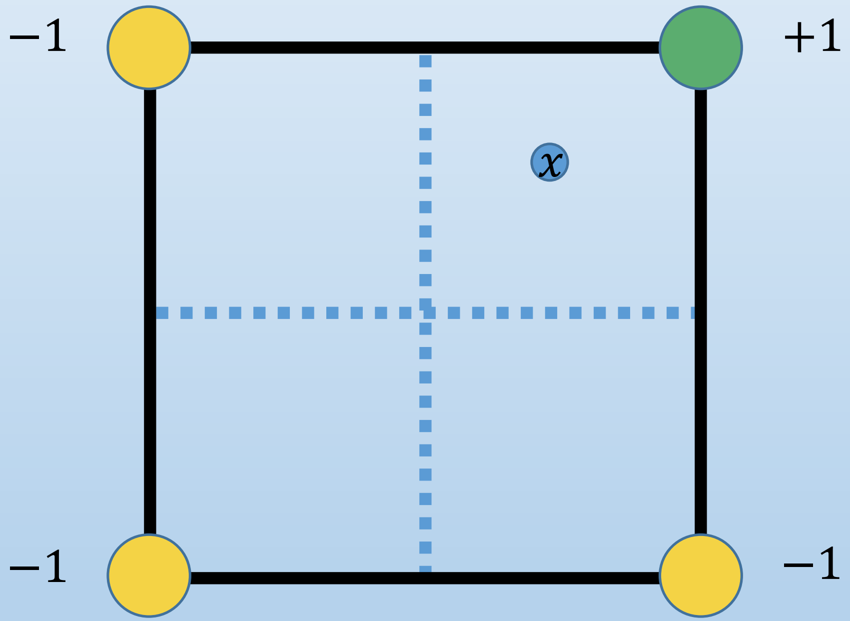

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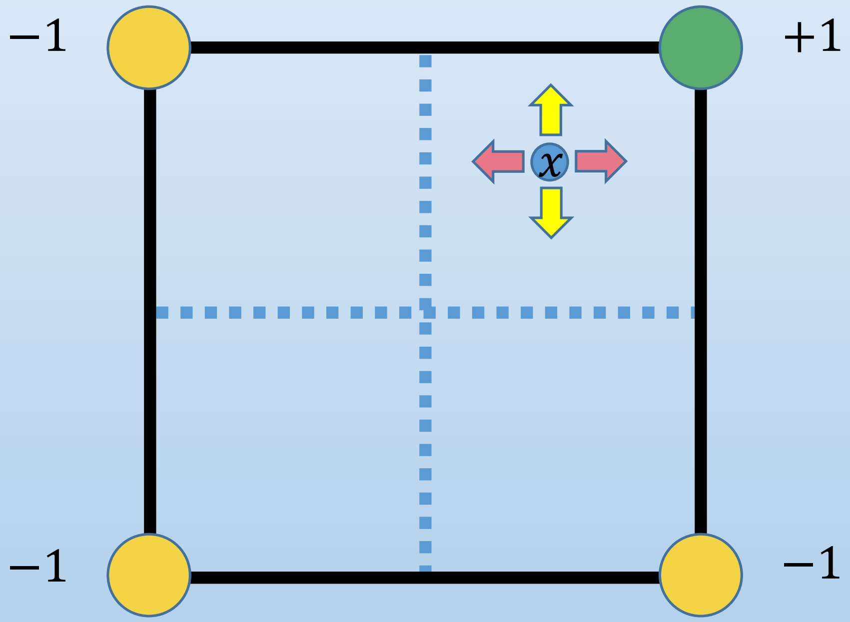

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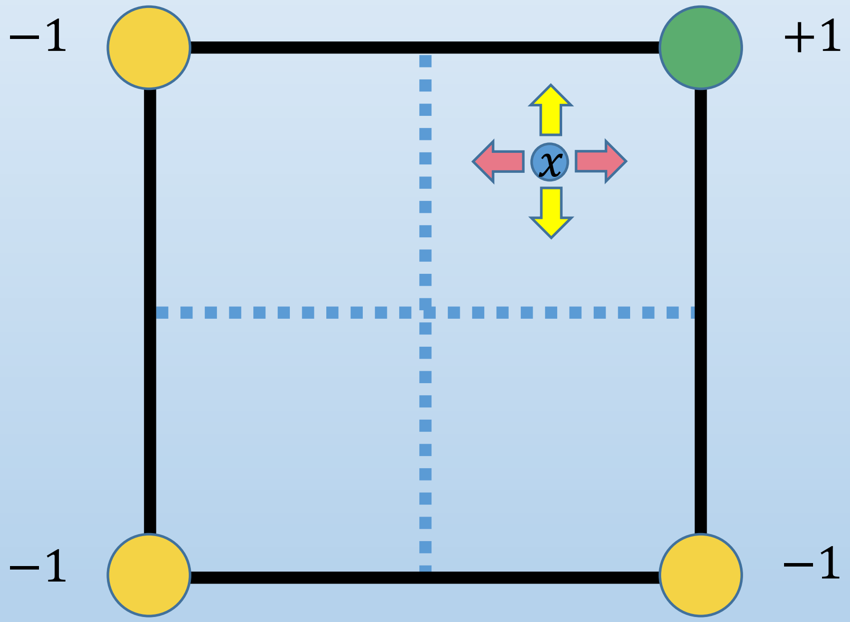

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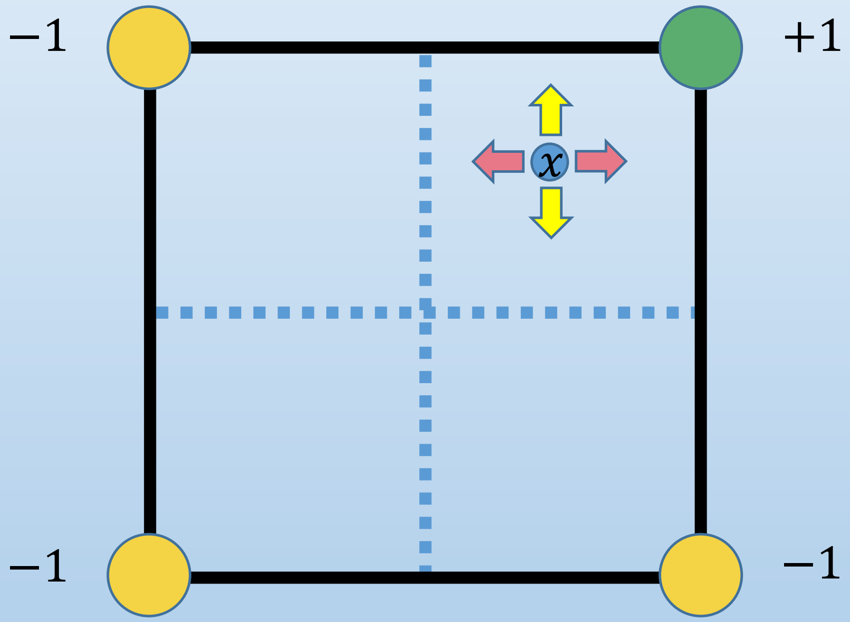

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- "Axis-aligned Laplacian" equation.

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- "Axis-aligned Laplacian" equation.
- $u(0)$ might tell us something about δ!
  - Solving a PDE can give us noise-sensitivity.

The OR function

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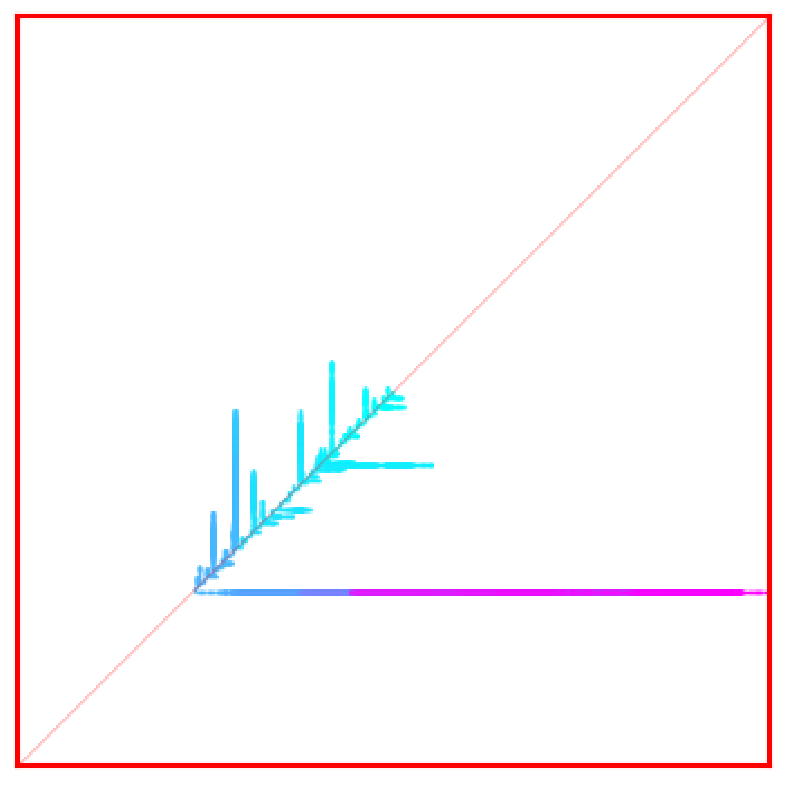

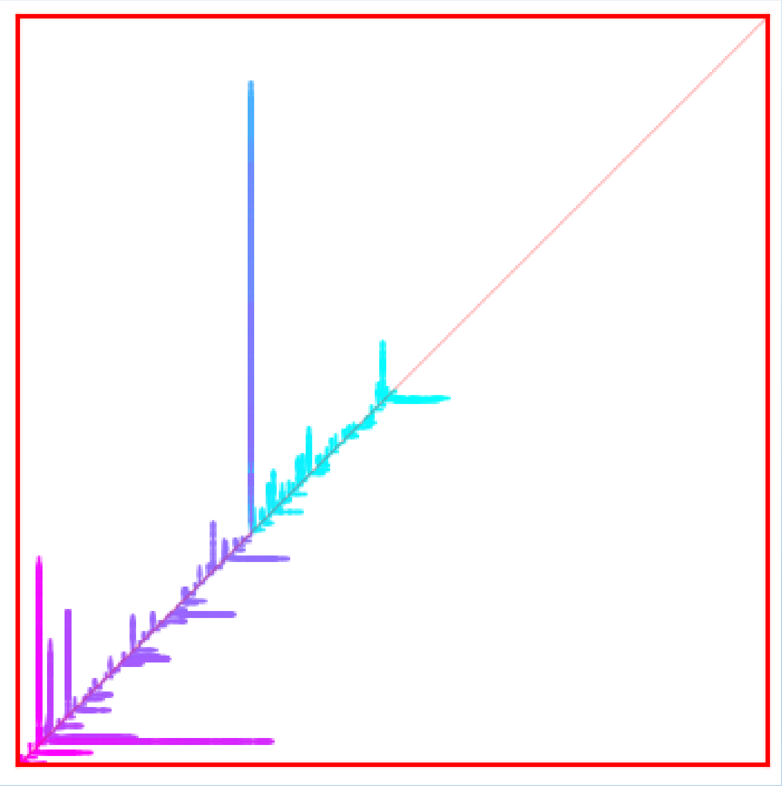

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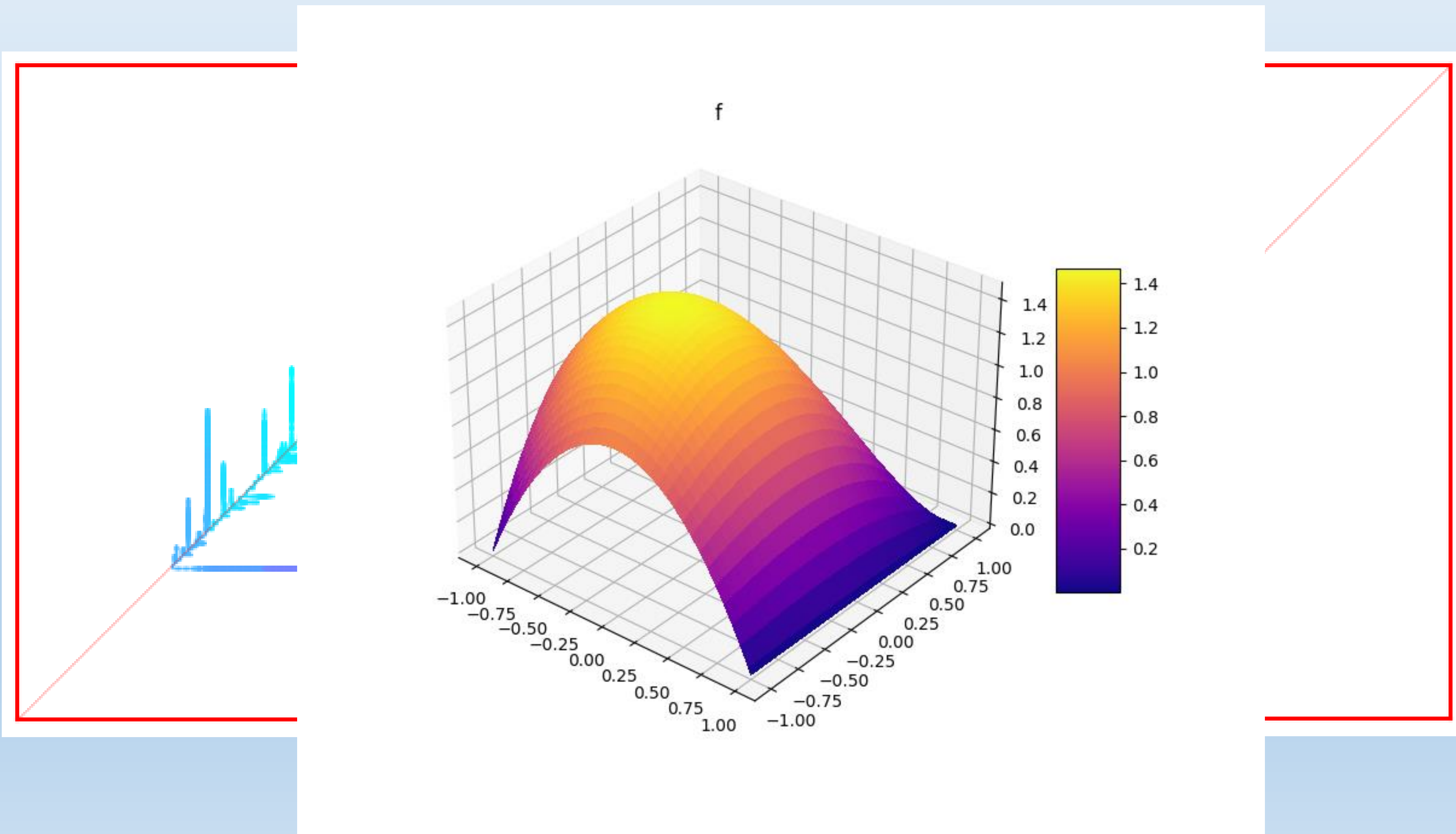

The big question

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- Is $P = NP$?

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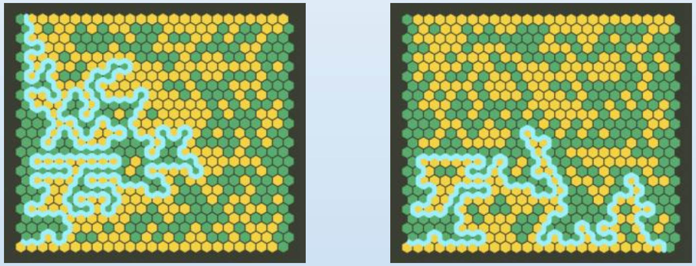
- Is there a class of functions f such that

$$\lim_{n\to\infty} \liminf_{\varepsilon\to 0} \frac{\delta(f,\varepsilon)}{\delta(f,1)} = 0?$$

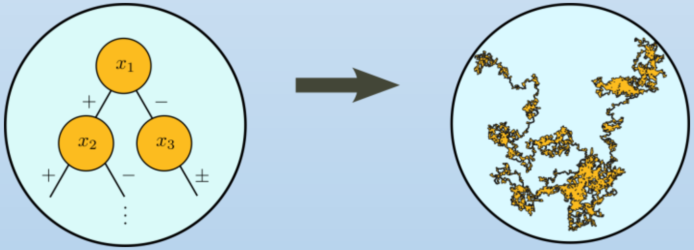
(specifically, what about percolation?)

Overview

- Boolean functions, noise-sensitivity, revealment algorithms

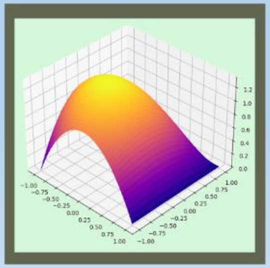


- Fractional algorithms can do better



- A limiting partial differential equation

$$\min_i \frac{\partial^2 u}{\partial x_i^2} = -2.$$

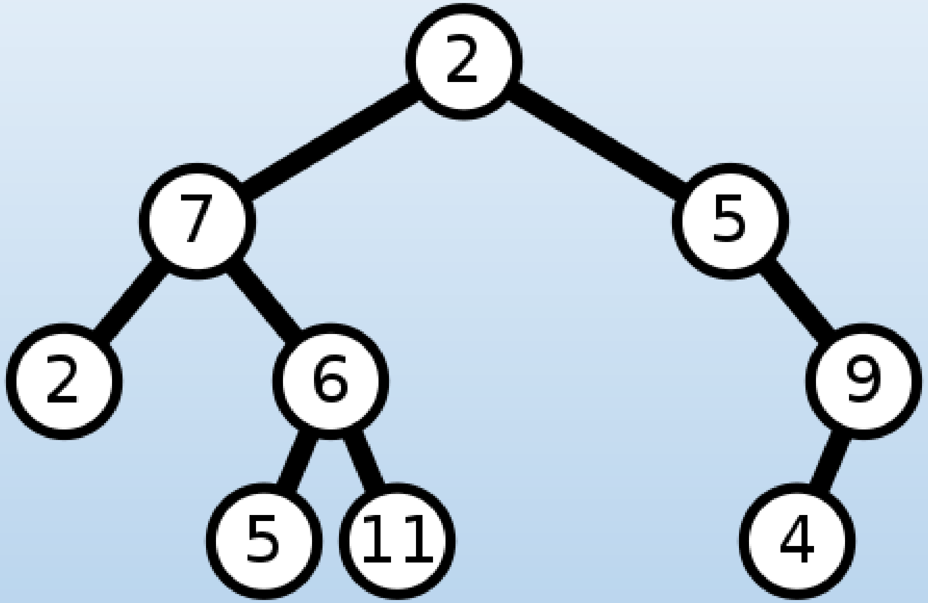


House

Person

Tree

https://practicalpie.com/house-person-tree-test/



Also a tree