A conformal Skorokhod embedding

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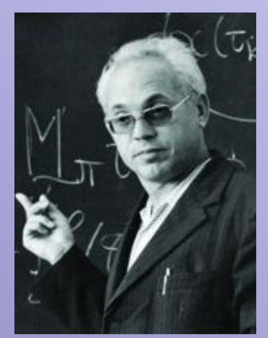
The problem

Skorokhod asks:

Q: Let μ be a centered distribution with finite variance, and let X_t be Brownian motion.

Can you find an integrable stopping time T so that $X_T \sim \mu$?



The theorem

For a domain $\Omega \subseteq \mathcal{R}^2$ and a planar Brownian motion

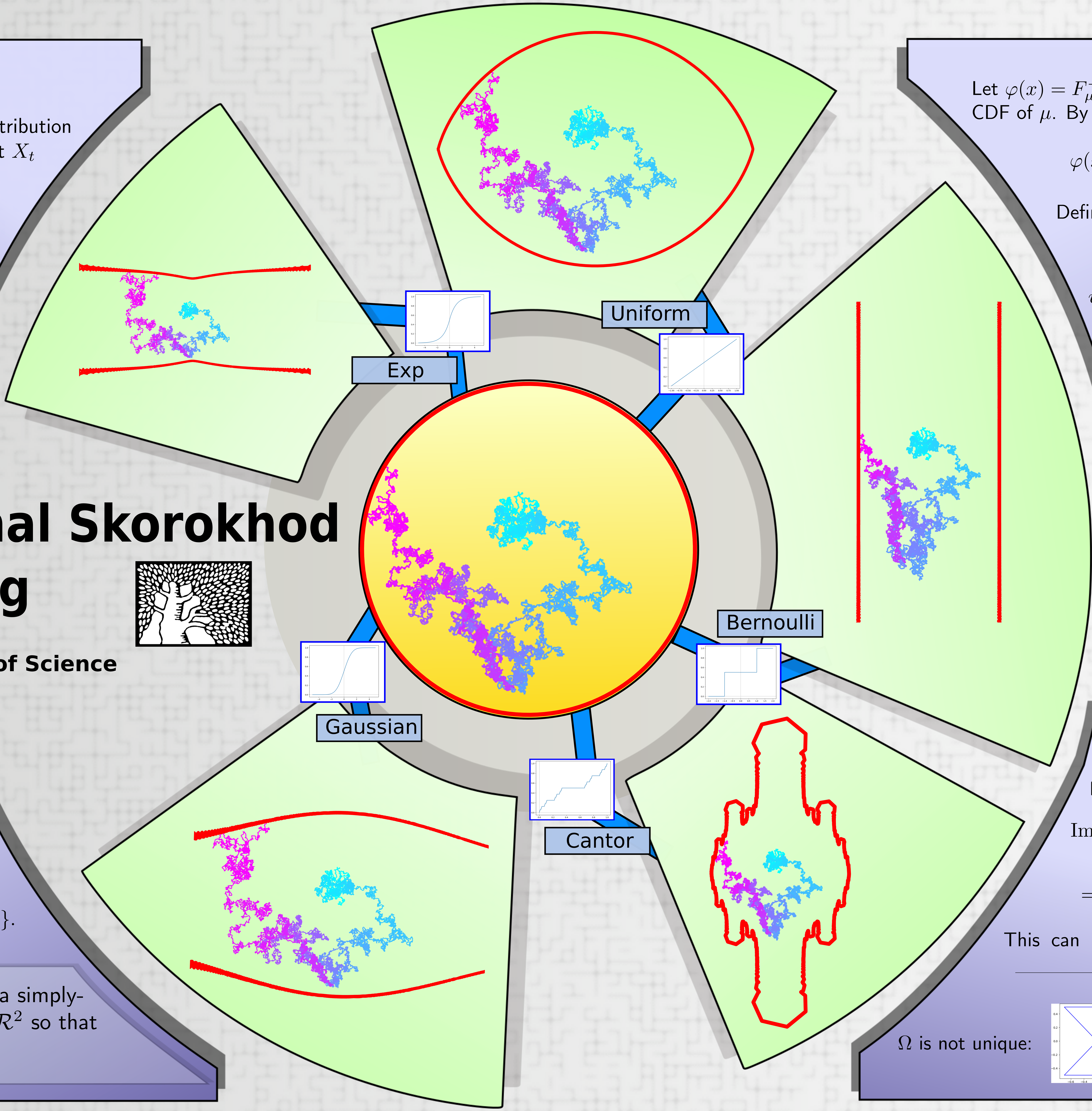
$$X_t = (X_t^1, X_t^2),$$

denote

$$T = \inf\{t > 0 \mid X_t \notin \Omega\}.$$

Theorem: There exists a simply-connected domain $\Omega \subseteq \mathcal{R}^2$ so that

$$X_T^1 \sim \mu.$$



The proof

Let $\varphi(x) = F_\mu^{-1}(|x|/\pi)$, where F is the CDF of μ. By Fourier decomposition,

$$\varphi(x) = \sum_{n=0}^{\infty} \widehat{\varphi}(n) \cos nx.$$

Define

$$\psi(z) = \sum_{n=0}^{\infty} \widehat{\varphi}(n) z^n.$$

ψ is conformal inside the unit disc (check winding numbers). By conformal invariance, $\psi(X_t)$ is Brownian motion, and

$$\psi(e^{i\theta}) = \varphi(\theta).$$

Now apply inverse sampling method.

The boundary

Boundary is given by the **Hilbert transform** of φ:

$$\mathrm{Im}\psi(e^{i\theta}) = \sum_{n=0}^{\infty} \widehat{\varphi}(n) \sin n\theta = \frac{1}{\pi} PV \int_{-\infty}^{\infty} \frac{\varphi(\tau)}{t-\tau} d\tau.$$

This can be used to calculate Ω.

Ω is not unique: