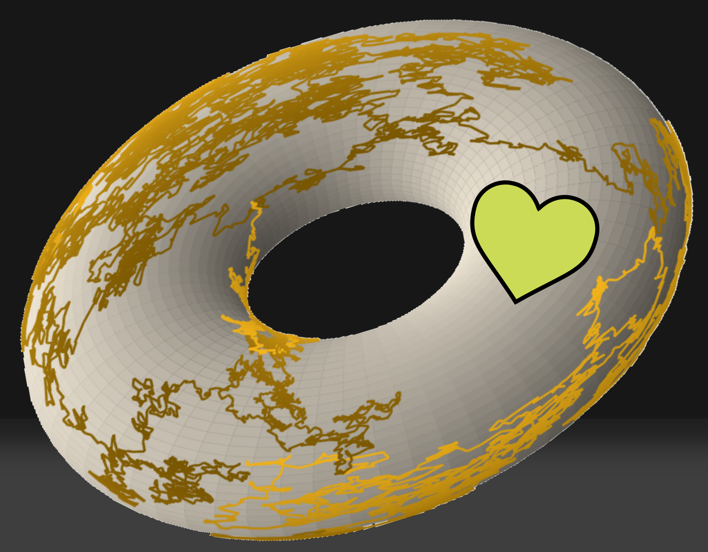


Brownian motion can feel the shape of a drum

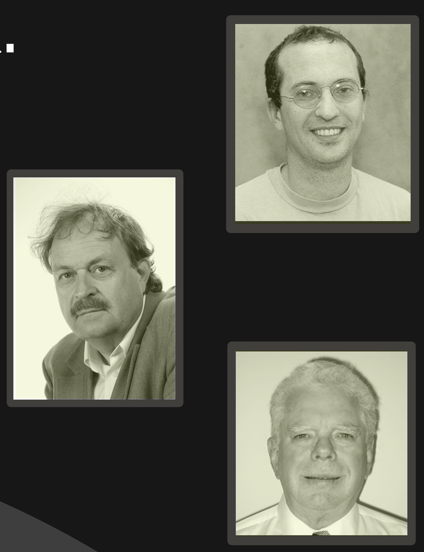
Renan Gross, Weizmann Institute of Science

THE PROBLEM

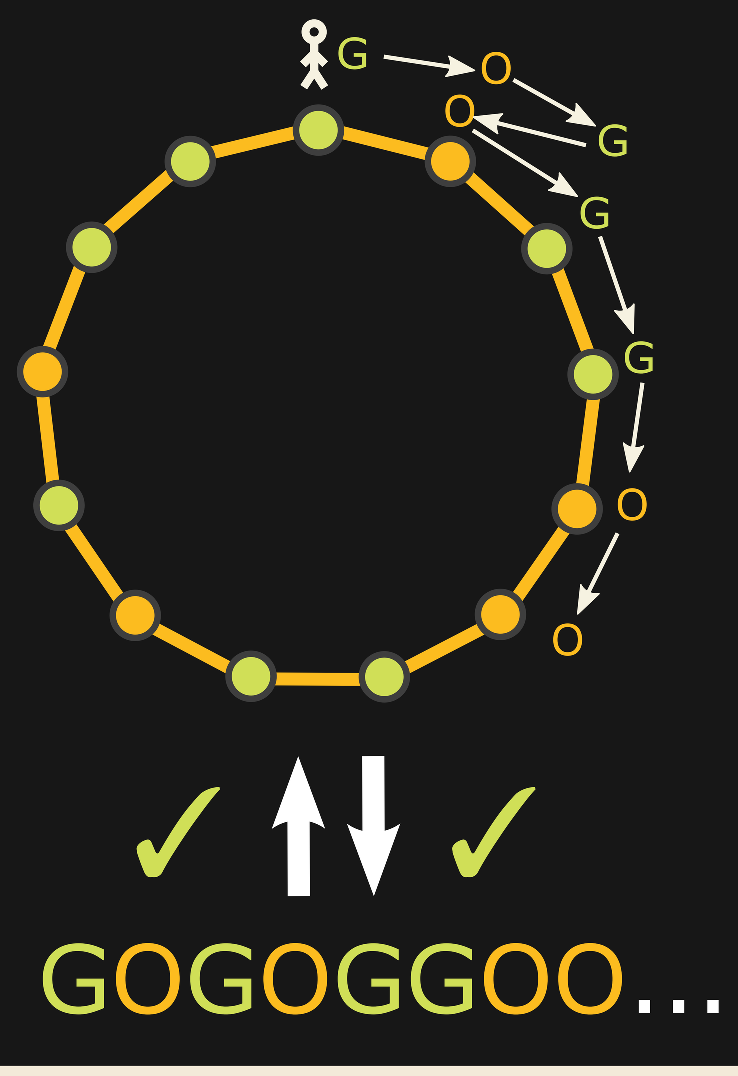
Inspired by Benjamini, Hollander and Keane, we ask:

Q: Let X_t be a stochastic process on $\mathbb{T}^d$, let A be a set, and let f be the indicator of A.

Can A be completely reconstructed from the (infinite) trace $f(X_t)$?



DISCRETE CASE



Color the cycle of length ℓ with two colors: G and O.

If the steps have distribution γ and the Fourier coefficients

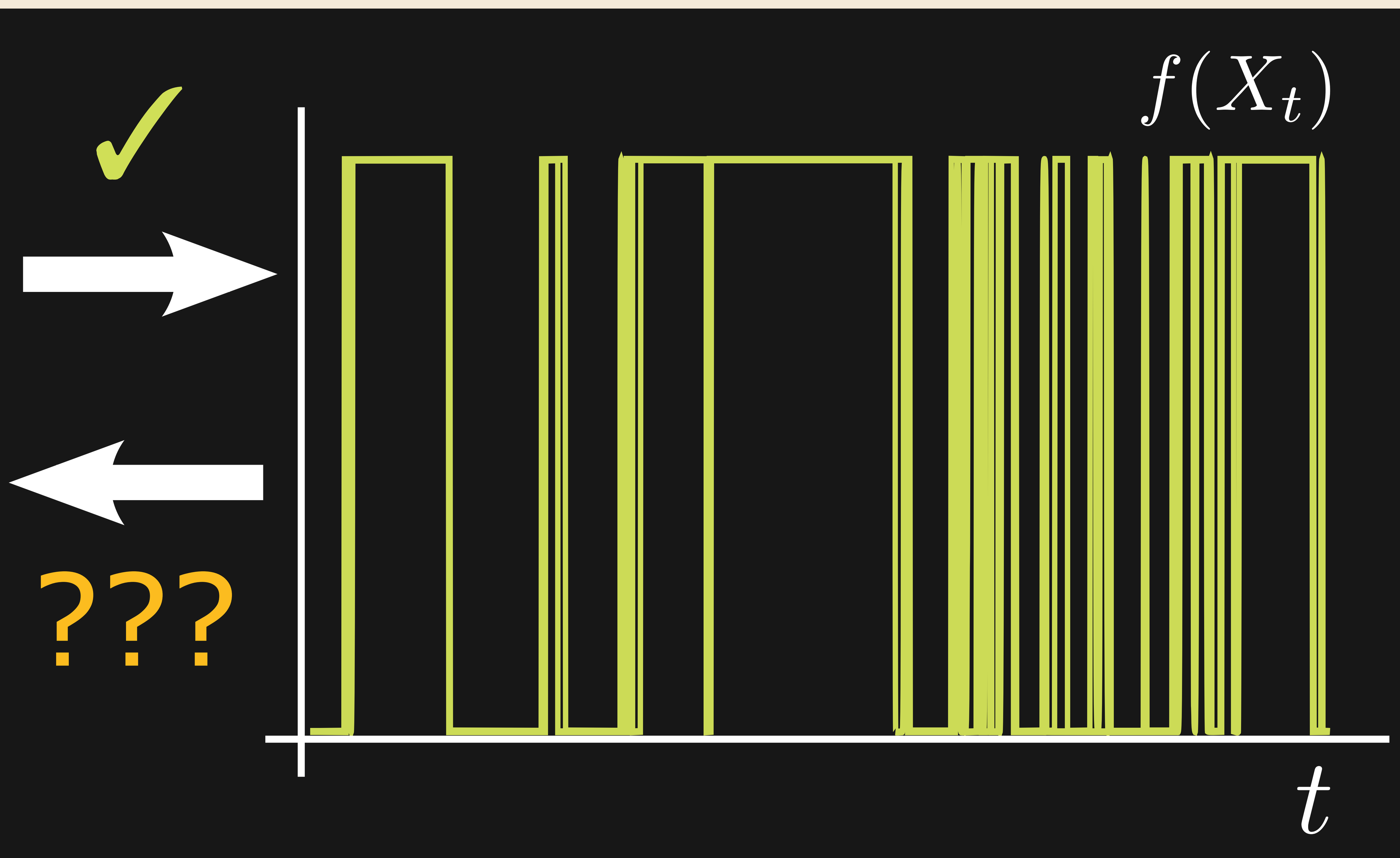
$$\{\hat{\gamma}(k)\}_{k=0}^{\ell-1}$$

are all distinct,

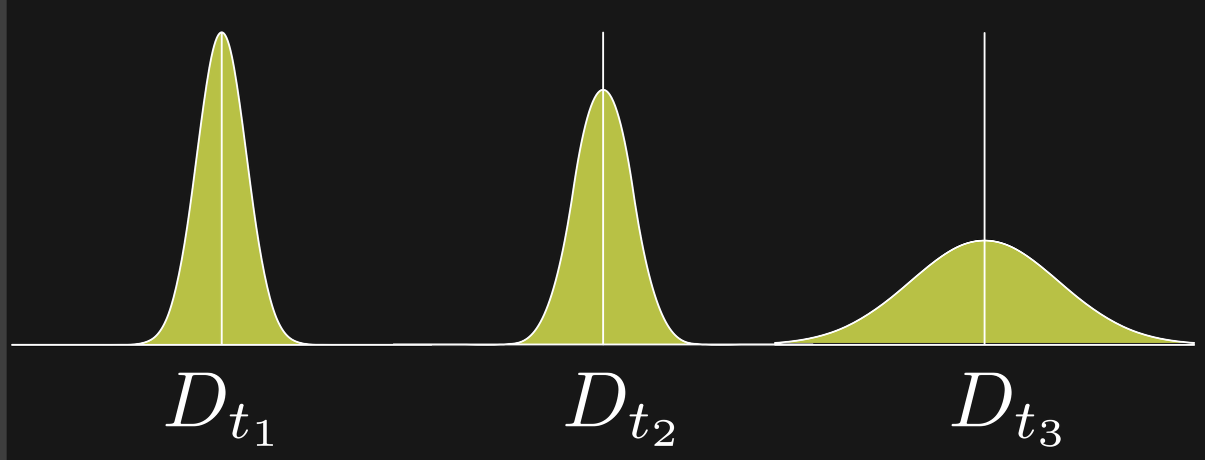
then the coloring can be reconstructed from the trace.

[Matzinger and Lember, 2006]

CONTINUOUS CASE



THE THEOREM



Let X_t be an infinitely divisible process on $\mathbb{T}^d$ whose increments D_t satisfy

$$D_t(x) = \beta_t \delta(x) + (1-\beta_t)\gamma_t(x),$$

where δ is the Dirac distribution and γ_t is an L^2 probability density function.

Theorem: If the Fourier coefficients $\{\hat{\gamma}_{t_0}\}_{k\in\mathbb{Z}^d}$ are all distinct and non-zero for some t_0, then A can be reconstructed from $f(X_t)$.

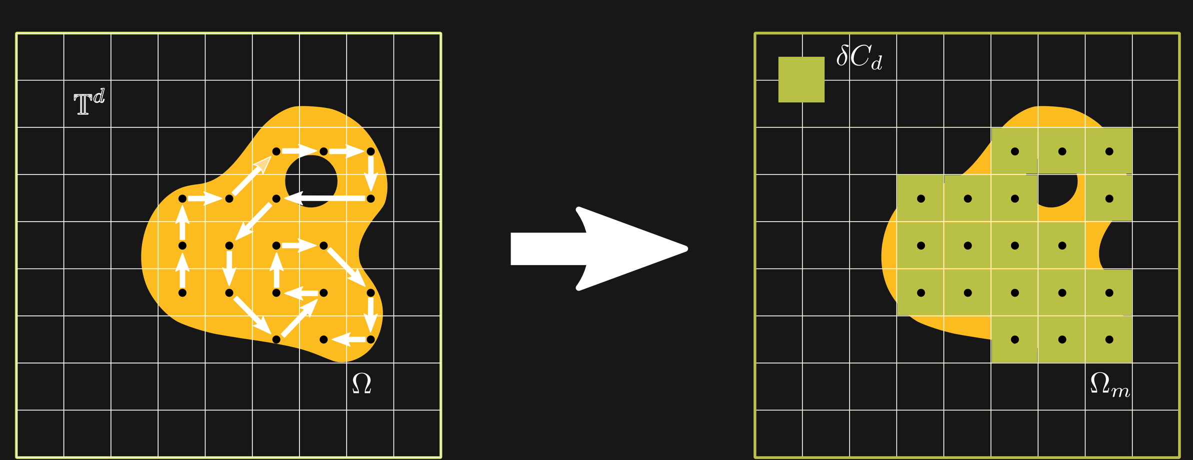
PROOF SKETCH

From the known temporal correlations

$$T_n(\mathbf{t}) = \mathbb{E}\left[f(X_0)\, f(X_{t_1}) \cdots f\left(X_{\sum_{i=1}^n t_i}\right)\right]$$

we can learn the spatial correlations

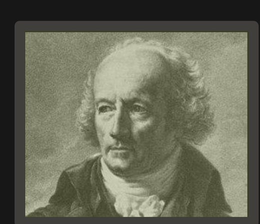
$$S_n(\mathbf{y}) = \int_{\mathbb{T}^d} f(x)\, f(x+y_1) \cdots f\left(x + \sum_{i=1}^n y_i\right) dx$$



∞ — VANDERMONDE

By applying Parseval's theorem on T_n, we can learn

$$\sum_{\mathbf{k}\in\mathbb{Z}^{nd}} \left(\prod_{i=1}^n \hat{\gamma}_{t_i}(k_i)\right)^m \hat{S}(\mathbf{k})$$



Now use the following lemma:

Lemma: Let $V_{ij} = z_j^i$, where $z_n \in \ell^2 \to 0,\ z_n \neq 0$.

If $x \in \ell^2$ s.t. $Vx = 0$,

then $x = 0$.

$$\begin{bmatrix} z_1 & z_2 & z_3 & \cdots \\ z_1^2 & z_2^2 & z_3^2 & \cdots \\ z_1^3 & z_2^3 & z_3^3 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$