Getting lost while hiking in the Boolean wilderness

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Student Probability Day VI, 11/05/17

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Joint work with Uri Grupel

Warning

This presentation shows explicit images of graphs.

Viewer discretion is advised.

Scenery reconstruction

Suppose we are given a graph G,

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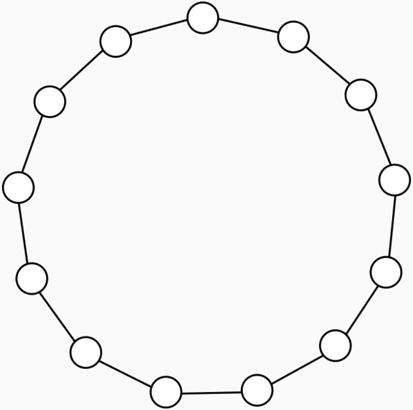
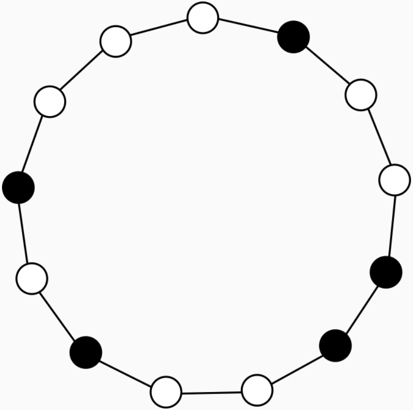


Scenery reconstruction

Suppose we are given a graph G, colored by some function $f(x)$.



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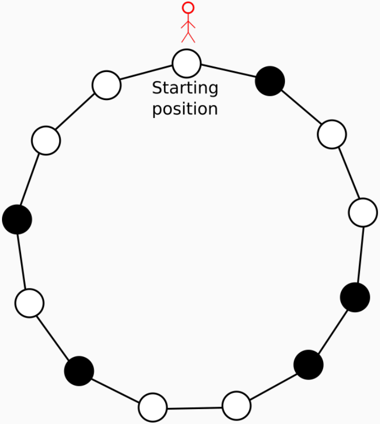

An agent performs a simple random walk S_t on the graph.

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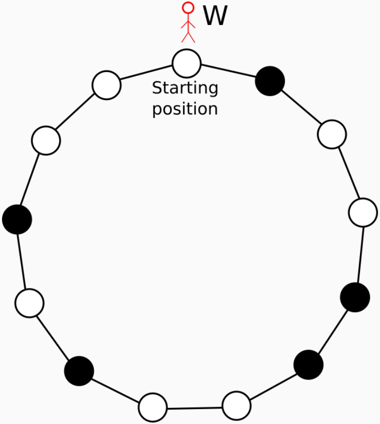

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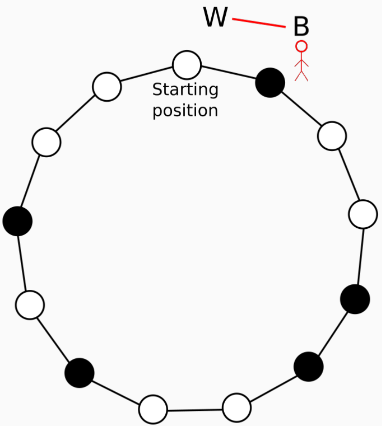
Reported scenery:

W

Scenery reconstruction



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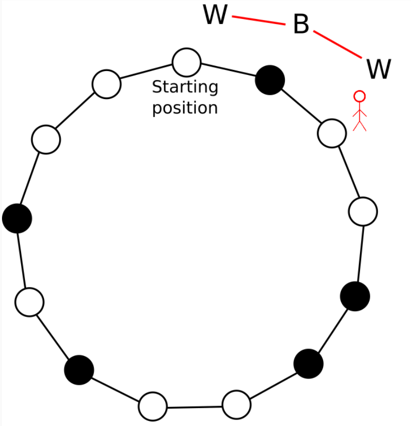

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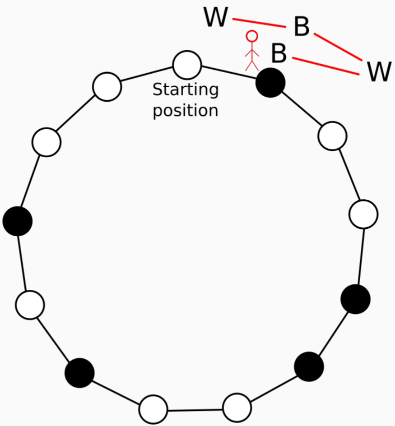

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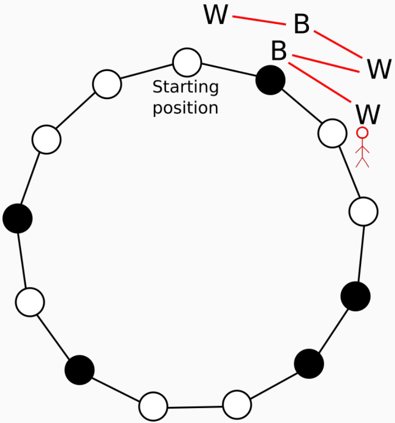

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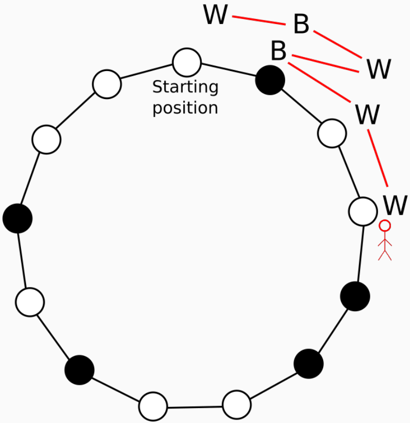

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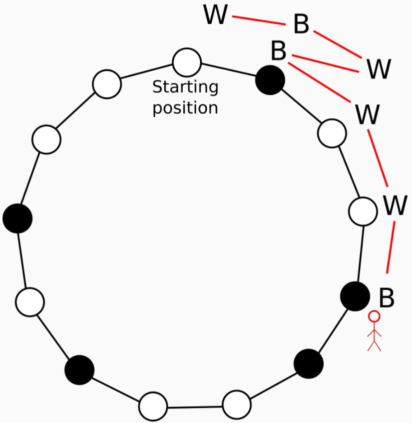

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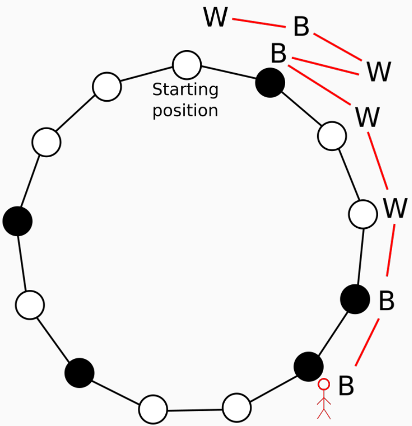

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Question

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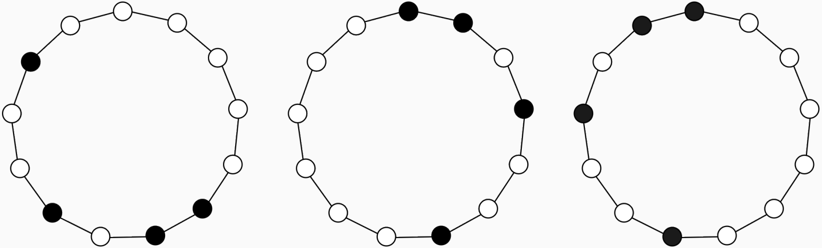
Remarks:

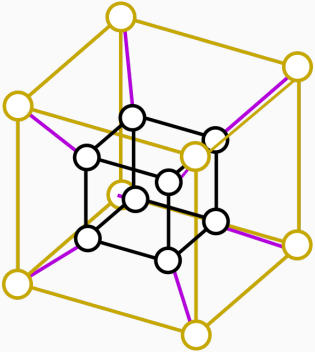
- We are given an infinite random walk $\rightarrow$ reconstruction should happen with probability 1
- Up to isomorphisms of the graph.



Scenery reconstruction

The answer is known for a variety of Abelian Cayley graphs and walks.

What about the *n*-dimensional Boolean hypercube?



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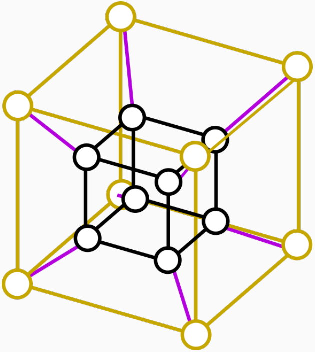

Vote?

Boolean scenery

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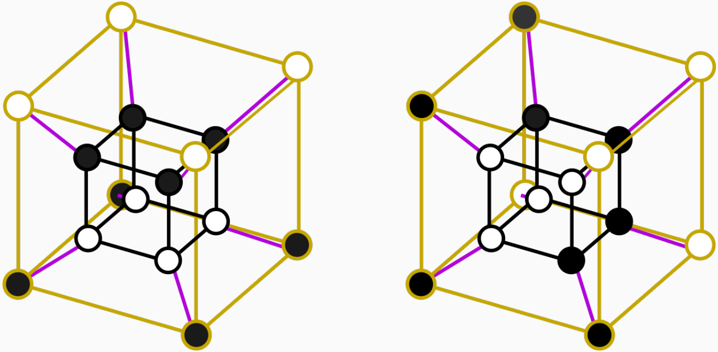

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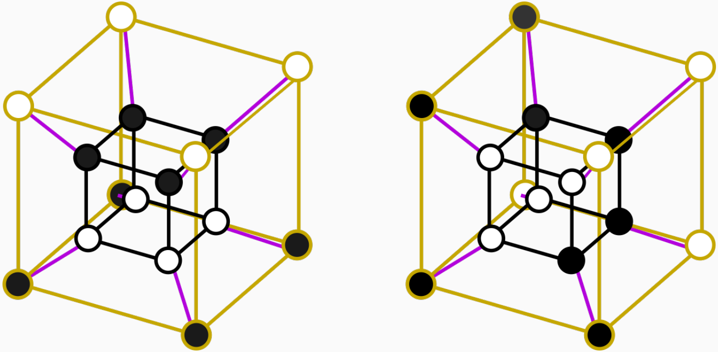

The process $f(S_t)$ is Bernoulli IID with success probability $1/2$!

Locally biased functions

So we defined locally biased functions, and started investigating them in their own right.

Definition

Let G be a graph. A Boolean function $f : G \to \{-1, 1\}$ is called *locally p-biased*, if for every vertex $x \in G$ we have

$$\frac{|\{y \sim x;\ f(y) = 1\}|}{deg(x)} = p.$$

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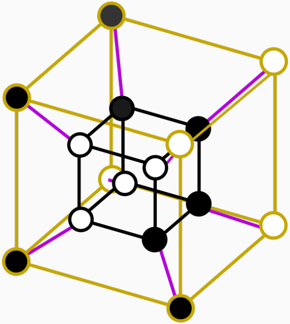
Existence of two non-isomorphic locally biased functions implies that the scenery reconstruction problem cannot be solved.

Locally biased functions

Locally biased functions can be defined for any graph.



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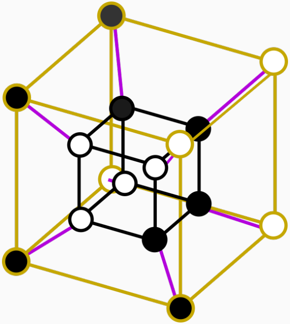

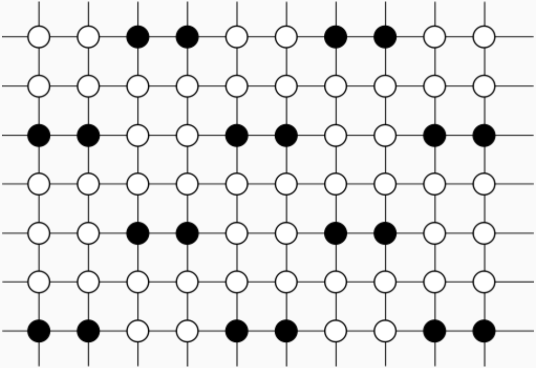

Theorems

We found:

Theorem (characterization)

Let $n \in \mathbb{N}$ be a natural number and $p \in [0,1]$. There exists a locally p-biased function $f : \{-1,1\}^n \to \{-1,1\}$ if and only if $p = b/2^k$ for some integers $b \geq 0, k \geq 0$, and 2^k divides n.

Theorem (size)

Let n be even. Let $B^n_{1/2}$ be a maximal class of non-isomorphic locally $1/2$-biased functions, i.e every two functions in $B^n_{1/2}$ are non-isomorphic to each other. Then $\left|B^n_{1/2}\right| \geq C2^{\sqrt{n}}/n^{1/4}$, where $C > 0$ is a universal constant.

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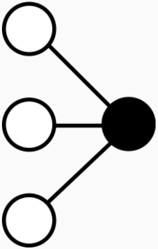

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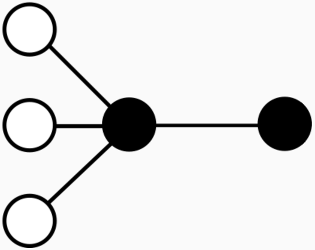

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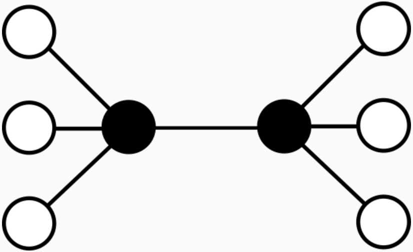

A tile for $n = 4$.

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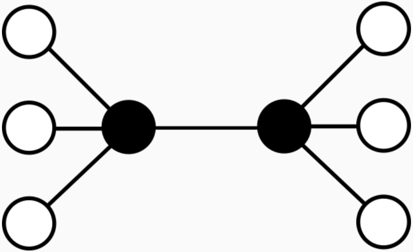

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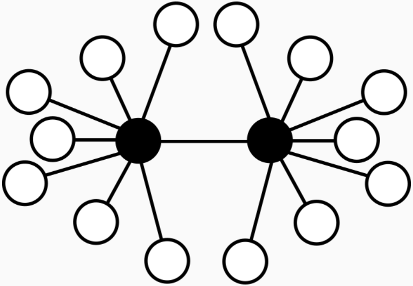
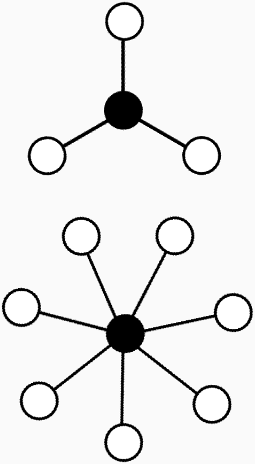


A tile for $n = 8$.

Proof of theorem 1

Solution: find a "half-tiling".



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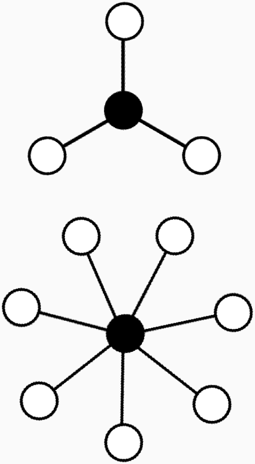

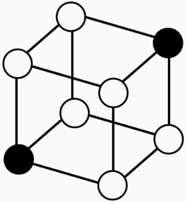

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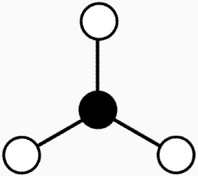

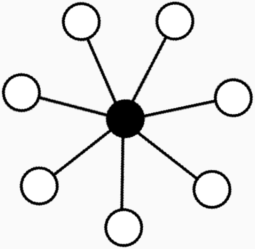

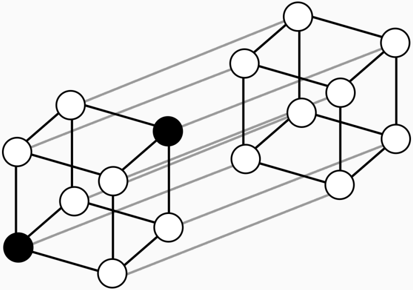

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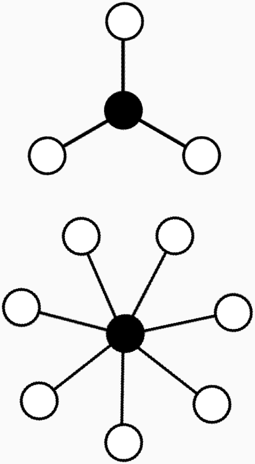

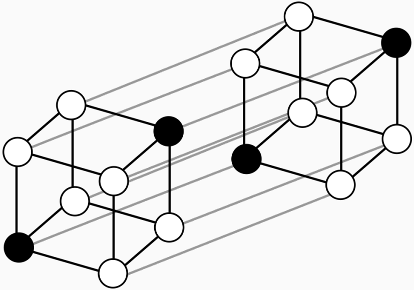

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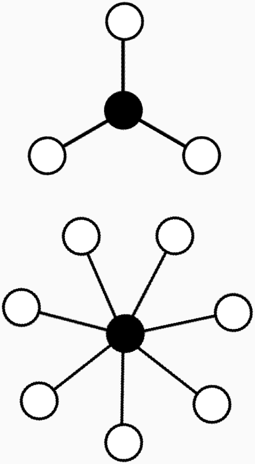
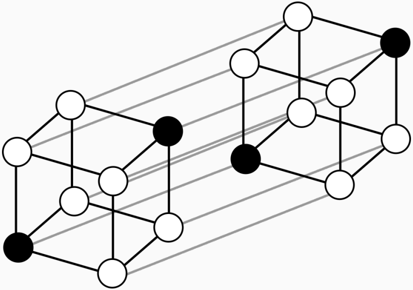
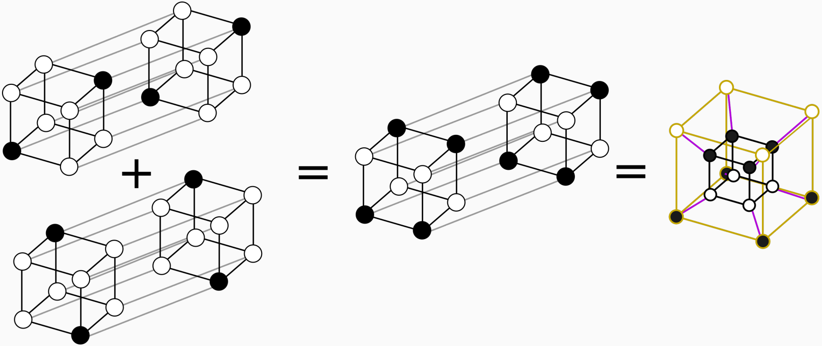




We have a $1/n$ tiling!

Proof of theorem 1

To get m/n instead of $1/n$, combine several disjoint tilings.



Proof of theorem 1

Where do we get half-tilings from? Perfect codes.

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Where do we get disjoint half-tilings from? Hamming perfect codes.

A word on the other direction

Let x be a uniformly random element of the cube. Then $f(x) = 1$ with probability $I/2^n$, where $I = |\{x \in \{-1,1\}^n\,;\ f(x) = 1\}|$

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Since both x and y are uniform random vertices,

$$\mathbb{P}(f(x) = 1) = \mathbb{P}(f(y) = 1)$$

A word on the other direction

Let x be a uniformly random element of the cube. Then $f(x) = 1$ with probability $l/2^n$, where $l = |\{x \in \{-1,1\}^n ;\ f(x) = 1\}|$

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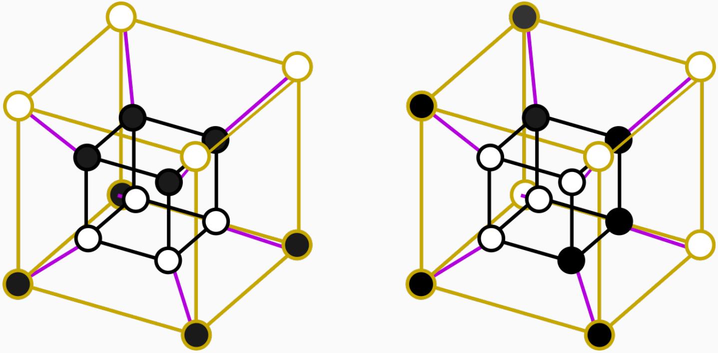
Denoting $p = m/n$ for some $m \in \{0, 1, \ldots, n\}$, this gives

$$p = \frac{l}{2^n} = \frac{m}{n}.$$

Writing $n = c2^k$ gives the desired result.

Proof of theorem 2



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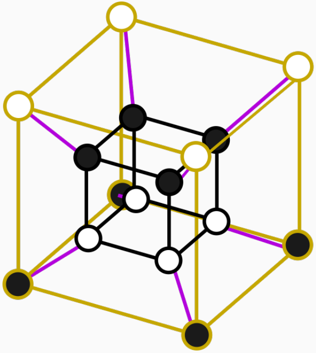

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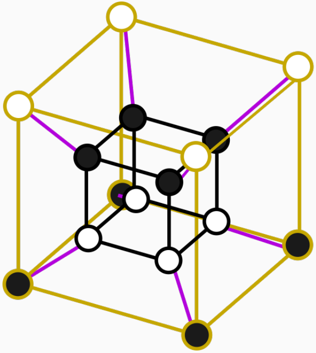
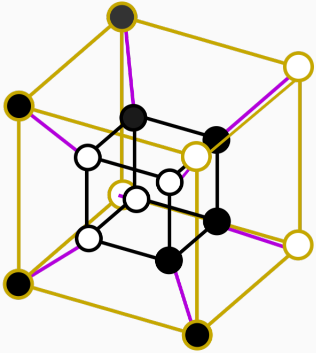


$g(x_1, x_2, x_3, x_4) = x_1x_2$



$h(x_1, x_2, x_3, x_4) =$
$\frac{1}{2}\left(x_1x_2 + x_2x_3 - x_3x_4 + x_1x_4\right)$

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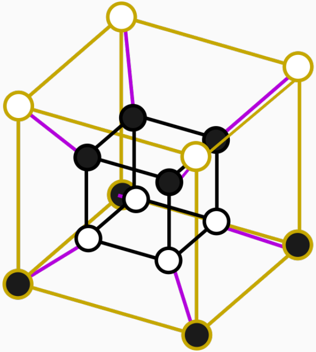

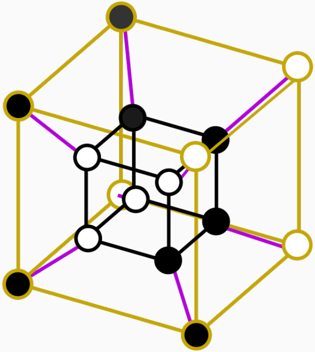

$$g_n(x_1, \ldots, x_n) = x_1 \cdots x_{n/2}$$

$$h_k = h\left(\prod_{i=0}^{k-1} x_{1+4i}, \ldots, \prod_{i=0}^{k-1} x_{4+4i}\right)$$

Proof of theorem 2

Fact

Let $f_i : \{-1, 1\}^{n_i} \to \{-1, 1\}$ *be locally* $1/2$*-biased functions for* $i = 1, 2$ *where* $n_1 + n_2 = n$. *Then*

$$f(x) = f_1(x_1, ..., x_{n_1}) f_2(x_{n_1+1}, ..., x_n)$$

is a locally $1/2$*-biased function on* $\{-1, 1\}^n$.

We can then build up locally $1/2$-biased functions from the building blocks $h_0, h_1, \ldots$ and $g_0, g_2, g_4, \ldots$.

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The # of different combinations is the same as the # of solutions to:

$$4a_1 + 8a_2 + \cdots + 4ka_k \leq n$$

which is at least

$$C \cdot 2^{\sqrt{n}}/n^{1/4}$$

for some constant C.

The end of the presentation

- There are other methods of showing that cube scenery cannot be reconstructed.
- But locally biased functions are still cool!
- Try them out for your favorite graphs!